Elliptic algebra of Holod and its generalizations

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1 Algebra of Holod

Petro Holod has proposed in 1983 (published in 1984) a new infinite-dimensional Lie algebra with the following basic elements:

$$S_\alpha^l = \lambda^l \sqrt{\lambda + a_\alpha} X_\alpha, \tag{1}$$

$$T_\alpha^m = \lambda^m \frac{\sqrt{\lambda + a_1}\sqrt{\lambda + a_2}\sqrt{\lambda + a_3}}{\sqrt{\lambda + a_\alpha}} X_\alpha, \tag{2}$$

where $\alpha \in \overline{1,3}, l, m \in \mathbb{Z}$, X_α is a basis of $so(3)$:

$$[X_\alpha, X_\beta] = \epsilon_{\alpha\beta\gamma} X_\gamma.$$

These basic elements satisfy the following relations:

$$[T_\alpha^l, T_\beta^m] = \epsilon_{\alpha\beta\gamma}(T_\gamma^{l+m+1} + a_\gamma T_\gamma^{l+m}) \tag{3a}$$

$$[T_\alpha^l, S_\beta^m] = \epsilon_{\alpha\beta\gamma}(S_\gamma^{l+m+1} + a_\beta S_\gamma^{l+m}) \tag{3b}$$

$$[S_\alpha^l, S_\beta^m] = \epsilon_{\alpha\beta\gamma} T_\gamma^{l+m}. \tag{3c}$$

Remark. Note that the algebra of Holod possess an "even" subalgebra spanned over the basic elements $T_\alpha^m, \alpha \in \overline{1,3},\ l, m \in \mathbb{Z}$.

The importance of the algebra of Holod in the theory of integrable systems is explained by the fact that it admits the Adler-Kostant-Symes decomposition into a direct sum of two Lie subalgebras:

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_{+} + \widetilde{\mathfrak{g}}_{-}. \tag{4}$$

This property permits one to obtain a wide set of commuting functions by restriction the invariants of the coadjoint representation of $\widetilde{\mathfrak{g}}$ onto the dual spaces of the subalgebras $\widetilde{\mathfrak{g}}_{\pm}$.

In the case of the algebra of Holod the subalgebras $\widetilde{\mathfrak{g}}_{\pm}$ are defined as follows:

$$\widetilde{\mathfrak{g}}_{+} = \mathrm{Span}_{\mathbb{C}}\{S^{l}_{\alpha},\ T^{m}_{\alpha}, \alpha \in \overline{1,3},\ l, m \geq 0\}, \tag{5}$$

$$\widetilde{\mathfrak{g}}_{-} = \mathrm{Span}_{\mathbb{C}}\{S^{l}_{\alpha},\ T^{m}_{\alpha}, \alpha \in \overline{1,3},\ l, m < 0\}. \tag{6}$$

Remark. Observe that the similar AKS decomposition has also the “even” subalgebra of the algebra of Holod. In this case one has that

$$\widetilde{\mathfrak{g}}_{+} = \mathrm{Span}_{\mathbb{C}}\{T^{m}_{\alpha}, \alpha \in \overline{1,3},\ m \geq 0\},$$
$$\widetilde{\mathfrak{g}}_{-} = \mathrm{Span}_{\mathbb{C}}\{T^{m}_{\alpha}, \alpha \in \overline{1,3},\ m < 0\}.$$

Let us remind that the following definition (Krichiver, Novikov 1986):

Definition. An infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ is called $\mathbb{Z}$-quasigraded of type (p, q) if it admits the decomposition:

$$\widetilde{\mathfrak{g}} = \sum_{j \in \mathbb{Z}} \mathfrak{g}_j, \quad \text{such that} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \sum_{k=-p}^{q} \mathfrak{g}_{i+j+k}.$$

It is evident that the algebra of Holod is quasigraded, but the type of the quasigrading will depend on how to define the spaces $\mathfrak{g}_j$.

In 1999 (published 2000) I have made the following important observation:

Proposition Let $\widetilde{\mathfrak{g}}$ be $\mathbb{Z}$-quasigraded Lie algebra of type $(0, 1)$. Then it admits the AKS decomposition:

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \widetilde{\mathfrak{g}}_-, \text{ with } \widetilde{\mathfrak{g}}_+ = \sum_{j \geq 0} \mathfrak{g}_j, \; \widetilde{\mathfrak{g}}_- = \sum_{j < 0} \mathfrak{g}_j.$$

Remark. The even subalgebra of the Holod's algebra is also quasigraded. It is of the type $(0, 1)$ with $\mathfrak{g}_j \simeq \mathbb{R}^3$.

2 The first — “irrational” generalization

The first generalization of the Holod’s algebra has several geometric interpretations.

2.1 The first generalization and higher genus curves

The first generalization of Holod’s algebra was found by me in 1999 (published by us in 2000).

In more details let us consider the classical matrix Lie algebras $\mathfrak{g} = gl(n)$, $so(n)$ and $sp(n)$. Let X_{ij} $i, j \in \overline{1, n}$ be their matrix basis.

Let a_i, b_i, $i \in \overline{1, n}$ be arbitrary complex parameters. Let us consider the following irrational “monomials” with the values in $\mathfrak{g}$:

$$X_{ij}^m = \lambda^m \sqrt{b_i \lambda + a_i} \sqrt{b_j \lambda + a_j} X_{ij}.$$

Using the commutation relations in the algebras $\mathfrak{g}$ it is possible to show that the elements X_{ij}^m, $i, j \in \overline{1, n}$, $m \in \mathbb{Z}$ span Lie algebras.

These “irrational” algebras may be viewed as the algebras of meromorphic functions on the (ramified) coverings of the hyperelliptic curve.

In the case of the $gl(n)$ the corresponding commutation relations have the form:

$$[X^p_{ij}, X^q_{kl}] = b_i\delta_{kj}X^{p+q+1}_{il} - b_j\delta_{il}X^{p+q+1}_{kj} + \\ + a_j\delta_{kj}X^{p+q}_{il} - a_i\delta_{il}X^{p+q}_{kj}.$$

In the case of the $so(n)$ the commutation relations are written as follows:

$$[X^p_{ij}, X^q_{kl}] = \\ b_j\delta_{kj}X^{p+q+1}_{il} - b_i\delta_{il}X^{p+q+1}_{kj} + b_j\delta_{jl}X^{p+q+1}_{ki} - b_i\delta_{ki}X^{p+q+1}_{jl} \\ + a_j\delta_{kj}X^{p+q}_{il} - a_i\delta_{il}X^{p+q}_{kj} + a_j\delta_{jl}X^{p+q}_{ki} - a_i\delta_{ki}X^{p+q}_{jl}$$

The algebra of Holod correspond to $\mathfrak{g} = so(4)$. The identification of the elements is following:

$$S^l_i = X^l_{4i},\ T^m_i = X^m_{jk}, \qquad (7)$$

where $b_1 = b_2 = b_3 = 1, b_4 = 0, a_4 = 1$.

The "even" subalgebra of Holod correspond in this picture to $\mathfrak{g} = so(3)$. The identification of the basic elements is the following:

$$T^m_i = X^m_{jk},\ \text{where } b_1 = b_2 = b_3 = 1. \qquad (8)$$

2.2 The first generalization and Lie pencils

The generalization of the Holod's algebra is connected also with the so-called "Lie pencils" or pairs of compatible Lie brackets.

Definition. Two Lie brackets $[\ ,\ ]_0$, $[\ ,\ ]_1$ on the linear space $\mathfrak{g}$ are called compatible if their arbitrary combination:

$$[\ ,\ ]_\lambda = [\ ,\ ]_0 + \lambda[\ ,\ ]_1$$

is again a Lie bracket.

So the first idea is to "affinize" the algebra $\mathfrak{g}$ with the bracket $[\ ,\ ]_\lambda$, considering the same complex parameter λ of affinization as in the above brackets, i.e. to consider the space $\mathfrak{g} \otimes \mathrm{Pol}(\lambda, \lambda^{-1})$ with brackets $[\ ,\ ]_\lambda$ (Skrypnyk 2002):

$$[X(\lambda), Y(\lambda)]_\lambda = [X(\lambda), Y(\lambda)]_0 + \lambda[X(\lambda), Y(\lambda)]_1.$$

(here $X(\lambda), Y(\lambda) \in \mathfrak{g} \otimes \mathrm{Pol}(\lambda, \lambda^{-1})$)

It is easy to see that the above-defined algebra is quasi-graded of the type $(0, 1)$ and, hence, admit AKS decomposition and may be used in the theory of integrable systems.

Let us now fix the compatible brackets on $\mathfrak{g}$, where $\mathfrak{g}$ is classical matrix Lie algebra, in the following way:

$$[X,Y]_0 = [X,Y]_A \equiv XAY - YAX,$$
$$[X,Y]_1 = [X,Y]_B \equiv XBY - YBX,$$

where A and B are constant matrices such that

$$[X,Y]_A \in \mathfrak{g},\ [X,Y]_B \in \mathfrak{g}, \forall X,Y \in \mathfrak{g}.$$

The corresponding algebra

$$\widetilde{\mathfrak{g}}_A \equiv (\mathfrak{g} \otimes \mathrm{Pol}(\lambda,\lambda^{-1}), [\ ,\ ]_{A(\lambda)})$$

is numbered by two numerical matrices A,B:

$$[\ ,\ ]_\lambda = [\ ,\ ]_{A+\lambda B}.$$

The algebra of Holod in this picture correspond to the case $\mathfrak{g} = so(4)$, and the diagonal matrices A and B:

$$A = \mathrm{diag}(a_1, a_2, a_3, 1),\ B = \mathrm{diag}(1,1,1,0).$$

The existence of the "hyperelliptic" or rather "irrational" realization of Holod's algebra is connected with the fact that the following relation for the algebra $(\mathfrak{g} \otimes \mathrm{Pol}(\lambda, \lambda^{-1}), [\ ,\ ]_\lambda)$ associated with the matrices A and B hold true:

$$[X(\lambda), Y(\lambda)]_{A(\lambda)} =$$
$$= [\sqrt{A(\lambda)} X(\lambda) \sqrt{A(\lambda)}, \sqrt{A(\lambda)} Y(\lambda) \sqrt{A(\lambda)}],$$

where $A(\lambda) = A + \lambda B$.

In such a realization the basis in the algebra

$$\widetilde{\mathfrak{g}}_A \equiv (\mathfrak{g} \otimes \mathrm{Pol}(\lambda, \lambda^{-1}), [\ ,\ ]_{A(\lambda)})$$

written with respect to the ordinary commutator consists of the irrational “monomials” of the following form:

$$X_{ij}^m = \lambda^m \sqrt{A(\lambda)} X_{ij} \sqrt{A(\lambda)},$$

which is in a full accord with the constructed above “hyperelliptic” or “irrational” realization.

2.3 The first generalization and classical r-matrices

In order to explain the relation of the material above with the theory of classical r-matrix we have to remind several definitions.

Definition 2. (Semenov 1983) The linear map R: $\widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ is called a classical R-operator if it satisfies "modified" Yang-Baxter equation:

$$R([R(X), Y] + [X, R(Y)]) - [R(X), R(Y)] = \\ = \frac{1}{4}[X, Y], \quad \forall X, Y \in \widetilde{\mathfrak{g}}. \quad (9)$$

The most important example of the classical R-operators is the so-called AKS R-operators:

$$R = \frac{1}{2}(P_{+} - P_{-}), \qquad (10)$$

which is known to satisfy modified classical Yang-Baxter equation (9) if $P_{\pm}$ are the projection operators onto the subalgebras $\widetilde{\mathfrak{g}}_{\pm}$ in AKS decomposition.

Let $\widetilde{\mathfrak{g}}$ be algebra of $\mathfrak{g}$-valued functions of one complex variable u. It is known (Semenov, Reyman 1988) that if the R-operator possesses a kernel i.e.:

$$R(X)(u_1) = \oint\limits_{u_2=0} (r_{12}(u_1, u_2), X_2(u_2))_2 \, du_2, \tag{11}$$

where $r_{12}(u_1, u_2)$ is the $\mathfrak{g} \otimes \mathfrak{g}$-valued function of two complex variables, $X_2 \equiv 1 \otimes X$, $(\ ,\ )$ invariant non-degenerated bilinear form on $\mathfrak{g}$, then the function $r_{12}(u_1, u_2)$ satisfies generalized classical Yang-Baxter equation (Maillet 1986):

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_3)] = [r_{23}(u_2, u_3), r_{12}(u_1, u_2)] - [r_{32}(u_3, u_2), r_{13}(u_1, u_3)], \tag{12}$$

$$r_{12}(u_1, u_2) \equiv \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} r^{\alpha\beta}(u_1, u_2) X_\alpha \otimes X_\beta \otimes 1 \text{ etc.}$$

In the case of skew-symmetric r-matrices when

$$r_{12}(u_1, u_2) = -r_{21}(u_2, u_1)$$

the equation (12) pass to the usual classical Yang-Baxter equation found by Sklyanin in 1979.

The constructed above algebra $\widetilde{\mathfrak{g}}_A$ of the irrational functions admit AKS decomposition, possess AKS R-operator and, hence, classical r-matrix. The direct calculation gives its following explicit form (T.Skrypnyk 2004):

$$r_A(\lambda,\mu) = \frac{1}{(\lambda-\mu)}\sum_{i,j=1}^{n} A(\lambda)^{1/2}X_{ij}A(\lambda)^{1/2}\otimes A(\mu)^{-1/2}X_{ji}A(\mu)^{-1/2}$$

In the case $A = \mathrm{diag}(a_1, ..., a_n)$, $B = \mathrm{diag}(b_1, ..., b_n)$ this formula acquires the form:

$$r_A(\lambda,\mu) = \frac{1}{(\lambda-\mu)}\sum_{i,j=1}^{n} \frac{\sqrt{(b_i\lambda + a_i)(b_j\lambda + a_j)}}{\sqrt{(b_i\mu + a_i)(b_j\mu + a_j)}} X_{ij}\otimes X_{ji}. \quad (13)$$

The discovered r-matrices are new: they are not skew-symmetric and are out of the Belavin-Drinfeld classification (Belavin-Drinfeld 1982).

By the other words, the third interpretation of the generalization of Holod's algebra is the following: it is an algebra associated with a certain class of non-skew-symmetric r-matrices.

2.4 List of new integrable systems associated with $\widetilde{\mathfrak{g}}_A$

1. Matrix and vector generalizations of anisotropic Heisenberg magnet equations:

$$\frac{\partial L}{\partial t} = [L, \frac{\partial^2 L}{\partial x^2}] + \frac{1}{2}\frac{\partial}{\partial x}(AL + LA) + \\ + [L, [L, \frac{\partial L}{\partial x}]]_A + \frac{1}{2}[L, AL + LA]_A,$$

2. Vector generalizations of Landau-Lifshitz hierarchy:

$$\frac{\partial \overrightarrow{s}}{\partial t} = \frac{\partial}{\partial x}\Big(\frac{\partial^2 \overrightarrow{s}}{\partial x^2} + 3/2(\frac{\partial \overrightarrow{s}}{\partial x}, \frac{\partial \overrightarrow{s}}{\partial x})\overrightarrow{s}\Big) + 3/2(\overrightarrow{s}, J\overrightarrow{s})\frac{\partial \overrightarrow{s}}{\partial x}, \tag{14}$$

where $J \equiv \mathrm{diag}(a_1^{-1}, a_2^{-1}, ..., a_{n-1}^{-1}, 0)$.

2. Matrix generalizations of the anisotropic chiral field equations:

$$\frac{\partial U}{\partial x_+} = [U, J_A(V)], \quad \frac{\partial V}{\partial x_-} = [V, J_A^{-1}(U)], \tag{15}$$

where $J_A(V) \equiv A^{1/2} V A^{1/2}$, $J_A^{-1}(U) \equiv A^{-1/2} U A^{-1/2}$.

3. Vector generalizations of the anisotropic

chiral field equations:

$$\partial_{x_+}\overrightarrow{s}_- = \Big(c_- - (\overrightarrow{s}_-, \overrightarrow{s}_-)\Big)^{1/2}\widehat{J}^{1/2}\overrightarrow{s}_+,$$

$$\partial_{x_-}\overrightarrow{s}_+ = \Big(c_+ - (\overrightarrow{s}_+, \overrightarrow{s}_+)\Big)^{1/2}\widehat{J}^{-1/2}\overrightarrow{s}_-,$$

where the $(n-2)\times(n-2)$ matrix $\widehat{J}$ is defined as follows: $\widehat{J} = \mathrm{diag}(a_2, ..., a_{n-1})$ and $c_\pm$ are arbitrary constants.

4. New Gaudin-type integrable systems:

$$H^{(k)} = \sum_{m=1}^{N}\frac{1}{(\nu_m-\nu_k)}\sum_{ij=1}^{n}\sqrt{\frac{(a_i+\nu_m)(a_j+\nu_m)}{(a_i+\nu_k)(a_j+\nu_k)}}S_{ij}^{(m)}S_{ji}^{(k)} + \sum_{ij=1}^{n}\frac{a_i}{(a_i+\nu_k)}S_{ij}^{(k)}S_{ji}^{(k)},\ \text{where}$$

in the case of $gl(n)$:

$$\{S_{ij}^{(p)}, S_{kl}^{(q)}\} = \delta_{pq}(\delta_{kj}S_{il}^{(q)} - \delta_{il}S_{kj}^{(q)}),$$

in the case of $so(n)$:

$$\{S_{ij}^{(p)}, S_{kl}^{(q)}\} = \delta_{pq}(\delta_{kj}S_{il}^{(p)} - \delta_{il}S_{kj}^{(p)} + \delta_{jl}S_{ki}^{(p)} - \delta_{ki}S_{jl}^{(p)}).$$

3 The second — “elliptic” generalization

3.1 Algebra of Holod as elliptic algebra

Let us consider again the basis in the algebra of Holod, written via the elliptic functions:

$$S^l_\alpha \equiv \lambda^l(u)\lambda_\alpha(u)\sigma_\alpha,$$
$$T^m_\alpha \equiv \lambda^l(u)\lambda_\beta(u)\lambda_\gamma(u)\sigma_\alpha,$$

where $\alpha, \beta, \gamma \in \overline{1,3},\ l, m \in \mathbb{Z}$,

$$\lambda(u) = \mathfrak{p}(u)$$

is a Weierstrass elliptic function, the functions $\lambda_\alpha(u)$ are defined via Jacobi elliptic functions:

$$\lambda_1(u) = \frac{1}{sn(u)},\ \lambda_2(u) = \frac{dn(u)}{sn(u)},\ \lambda_2(u) = \frac{cn(u)}{sn(u)} \tag{16}$$

and σ_α, $\alpha \in \overline{1,3}$ are the Pauli matrices constituting a basis in $sl(2) \simeq so(3)$.

This definition of the basis is consistent with a previous one by the virtue of the properties of Weierstrass and Jacobi functions. Moreover, it hints a way of another generalization of the Holod's algebra.

3.2 The elliptic generalization and θ-functions

Let $a \in Z_n \times Z_n$. Let us consider the meromorphic functions on the elliptic curve defined in the following way:

$$\lambda_a(u) = \frac{\sigma_a(u)\sigma_0'(0)}{\sigma_0(u)\sigma_a(0)} \tag{17}$$

Where $\sigma_a(u)$ — is the θ-function with the characteristics $(a_2/n+1/2, a_1/n+1/2)$, $a \neq 0 \bmod n\mathbb{Z} \times n\mathbb{Z}$:

$$\sigma_a(u) \equiv \theta_{(a_2/n+1/2, a_1/n+1/2)}(u, \tau) =$$
$$= \sum_{k \in Z} \exp(\pi i(a_2/n + 1/2 + k)^2\tau +$$
$$+ 2\pi i(a_2/n + 1/2 + k)(u + a_1/n + 1/2))$$

The functions $\lambda_a(u)$ have the following decomposition in the Laurent power series in a neighborhood of zero:

$$\lambda_a(u) = \frac{1}{u} + c_a + b_a u + d_a u^2 + g_a u^3 + \cdots , \quad (18)$$

where coefficients c_a, b_a, d_a, g_a etc. are expressed via θ-constants.

Let us consider the following example:

Example Let $n = 2$. In this case we have the following set of possible indices $\{(a_1, a_2)\} = \{(1,1), (0,1), (1,0)\}$. The corresponding elliptic functions $\lambda_a(u)$ could be simply written via the Jacobi elliptic functions:

$$\lambda_{(1,1)}(u) = \frac{dn(u)}{sn(u)},$$

$$\lambda_{(0,1)}(u) = \frac{1}{sn(u)},$$

$$\lambda_{(1,0)}(u) = \frac{cn(u)}{sn(u)}.$$

3.3 Special bases for $sl(n)$ algebra.

Let us consider a Lie algebra $\mathfrak{g} = sl(n, \mathbb{C})$. For any $a \in Z_n^2$, $a \neq (0,0)$ let us define matrix T_a:

$$T_a = T_{(1,0)}^{a_1} T_{(0,1)}^{a_2},$$

$$T_{(1,0)} = \begin{pmatrix} 1 & 0 & 0 & . & . & 0 \\ 0 & \epsilon & 0 & . & . & 0 \\ 0 & 0 & \epsilon^2 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & \epsilon^{n-1} \end{pmatrix},$$

$$T_{(0,1)} = \begin{pmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & 1 & . \\ 0 & . & . & . & 0 & 1 \\ 1 & 0 & 0 & . & . & 0 \end{pmatrix}.$$

It is known that the elements $\{T_a\}$ form a basis in $sl(n)$. Commutation relation have the form:

$$[T_a, T_b] = \alpha_{a,b} T_{a+b}, \qquad (19)$$

where $\alpha_{a,b} = \epsilon^{a_2 b_1} - \epsilon^{a_1 b_2}$, $\epsilon = e^{\frac{2\pi i}{n}}$.

Example Let $n = 2$. In this case the basic elements of $sl(2,\mathbb{C}) \simeq so(3,\mathbb{C})$ are:

$$T_3 \equiv T_{(1,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$T_2 \equiv \ T_{(0,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$T_1 \equiv T_2 T_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Commutation relations in the chosen basis are:

$$[T_1, T_2] = -2T_3, \ \ [T_2, T_3] = 2T_1, \ \ [T_1, T_3] = 2T_2.$$

Rescaling the generators $T_1 \to \frac{1}{2}T_1$, $T_2 \to \frac{i}{2}T_2$, $T_3 \to \frac{i}{2}T_3$ we obtain the standard $so(3)$ commutation relations:

$$[T_i, T_j] = -\epsilon_{ijk} T_k.$$

3.4 Elliptic $sl(n)$-valued Lie algebra.

Now we will define the elliptic Lie algebra $\mathfrak{E}(sl(n))$. For this purpose we introduce certain elliptic functions with the values in $sl(n)$:

$$Y_a^n(u) = T_a \otimes \lambda^n(u)\lambda_a(u),$$

$$X_a^m(u) = -T_a \otimes \lambda^m(u)\frac{d\lambda_a(u)}{du}.$$

By the direct calculations it is possible to show that they satisfy the following commutation relations (T. Skrypnyk 2012):

$$[X_a^n, X_b^m] = \alpha_{a,b}(X_{a+b}^{n+m+1} - (b_a + b_b - b_{a+b})X_{a+b}^{n+m} +$$
$$+ (2c_{b-a}(b_a - b_b) + (c_a + c_b)(b_{a-b} + b_{a+b}) - 2(c_a b_b + c_b b_a))Y_{a+b}^{n+m}$$

$$[X_a^n, Y_b^m] = \alpha_{a,b}(Y_{a+b}^{n+m+1} + (b_a + b_{a+b} - b_b)Y_{a+b}^{n+m} +$$
$$+ (c_{a+b} - c_b)X_{a+b}^{n+m})$$

$$[Y_a^n, Y_b^m] = \alpha_{a,b}(X_{a+b}^{n+m} + (c_a + c_b)Y_{a+b}^{n+m}).$$

Let us now define elliptic Lie algebra $\mathfrak{E}(sl(n))$ as a linear space spanned over the above basis:

$$\mathfrak{E}(sl(n)) \equiv Span_C\{X_a^l, Y_b^m; l, m \in Z, a, b \in Z_n \times Z_n.\}$$

The following theorem holds (T.Skrypnyk 2012):

Theorem

(i) Lie algebra $\mathfrak{E}(sl(n))$ admits AKS scheme

$$\mathfrak{E}(sl(n)) = \mathfrak{E}(sl(n))_+ + \mathfrak{E}(sl(n))_-,$$

$$\mathfrak{E}(sl(n))_+ \equiv Span_{\mathbb{C}}\{X_a^l, Y_b^m; l, m \geq 0, a, b \in Z_n \times Z_n.\},$$
$$\mathfrak{E}(sl(n))_- \equiv Span_{\mathbb{C}}\{X_a^l, Y_b^m; l, m < 0, a, b \in Z_n \times Z_n.\}$$

(ii) $\mathfrak{E}(sl(n))$ is $\mathbb{Z}$- quasi-graded Lie algebra of the type $(0,3)$ and the quasigrading is connected with AKS decomposition, i.e. its restriction to the subalgebras $\mathfrak{E}(sl(n))_\pm$ yields $\mathbb{Z}_\pm$ quasigrading of the algebras $\mathfrak{E}(sl(n))_\pm$.

(iii) Algebras $(\mathfrak{E}(sl(n)), \mathfrak{E}(sl(n))_+, \mathfrak{E}(sl(n))_-)$ constitute Manin triple:

$$\mathfrak{E}(sl(n))_+^* = \mathfrak{E}(sl(n))_-, \ \mathfrak{E}(sl(n))_-^* = \mathfrak{E}(sl(n))_+.$$

Example Let us consider in more details an $sl(2,\mathbb{C})$ automorphic elliptic algebra. In this case $c_a \equiv 0$, $b_{a+b} + b_a + b_b = 0$, $j_a = 2b_a$, (here j_a are the branching points of the elliptic curve) and we obtain for the algebra $\mathfrak{E}(sl(2))$ the following commutation relations:

$$[X_a^n, X_b^m] = \alpha_{a,b}(X_{a+b}^{n+m+1} + j_{a+b}X_{a+b}^{n+m})$$
$$[X_a^n, Y_b^m] = \alpha_{a,b}(Y_{a+b}^{n+m+1} + j_b Y_{a+b}^{n+m})$$
$$[Y_a^n, Y_b^m] = \alpha_{a,b}X_{a+b}^{n+m}.$$

As it follows from the example above, up to the renaming of variables and parameters of anisotropy it is exactly the algebra of Holod.

3.5 The second generalization and elliptic r-matrices

Due to the Theorem above the algebra $\mathfrak{E}(sl(n))$ admits AKS shcheme and, hence, possess the classical R-operators and classical r-matrix.

Using the results of the previous subsections and the addition formulas for theta-functions it is possible to prove the following theorem (T.Skrypnyk 2013):

Theorem. The kernel (as the kernel of the integral operator) of the classical AKS R-operator on $\mathfrak{E}(sl(n))$ coincide with the following classical r-matrix:

$$r(u,v) = \sum_{a \in Z_n \times Z_n} \lambda_a(u-v) T_a \otimes T_{-a}.$$

The r-matrix is skew-symmetric and coincide with elliptic r-matrix of Belavin (Belavin 1981).

4 The r-matrix algebras

The connection of the above generalizations of Holod's algebra with the classical r-matrices suggests its ultimate generalization (Skrypnyk 2013):

Theorem. With any classical r-matrix satisfying the generalized classical Yang-Baxter equation is possible to associate infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}_r$ with the following properties:

(i) AKS decomposition:

(ii) Z-quasigrading of the type $(0, q)$, where $q \geq 1$ and the quasigrading is compatible with AKS decomposition, i.e. its restriction to the subalgebras $\widetilde{\mathfrak{g}}_r^{\pm}$ yields $\mathbb{Z}_{\pm}$ quasigradings $\widetilde{\mathfrak{g}}_r^{\pm}$.

(iii) if the r-matrix is skew-symmetric then the algebras $(\widetilde{\mathfrak{g}}_r, \widetilde{\mathfrak{g}}_r^{+}, \widetilde{\mathfrak{g}}_r^{-})$ constitute Manin triple: $(\widetilde{\mathfrak{g}}_r^{\pm})^* = \widetilde{\mathfrak{g}}_r^{\mp}$.

Remark. By the other words the algebra of Holod is historically a second example of the r-matrix Lie algebras. The first example is a loop algebra corresponding to the rational r-matrix.

Thank you for the attention !