Integrable Dispersive Chains

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The General Problem. The Korteweg de Vries equation

- The Korteweg de Vries equation is associated with the linear Schrödinger equation

$$\psi_{xx} = (\lambda + u)\psi.$$

The function $\psi(x,t,\lambda)$ satisfies the pair of *linear* equations in partial derivatives

$$\psi_{xx} = u\psi, \quad \psi_t = a\psi_x - \frac{1}{2}a_x\psi.$$

Then the compatibility condition $(\psi_{xx})_t = (\psi_t)_{xx}$ yields the relationship

$$u_t = \left(-\frac{1}{2}\partial_x^3 + 2u\partial_x + u_x\right) a$$

between functions $u(x,t,\lambda)$ and $a(x,t,\lambda)$.

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between functions $u(x, t, \lambda)$ and $a(x, t, \lambda)$.
- If we choose the linear dependences $u(x, t, \lambda) = \lambda + u^1(x, t)$ and $a(x, t, \lambda) = \lambda + a_1(x, t)$, we obtain nothing but the famous Korteweg de Vries equation
$$u_t^1 = \frac{1}{4}u_{xxx}^1 - \frac{3}{2}u^1 u_x^1,$$

The General Problem. The Kaup–Boussinesq system

- Again we consider the relationship

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- If we choose the quadratic dependence $u(x, t, \lambda) = \lambda^2 + \lambda u^1(x, t) + u^2(x, t)$ and again the linear dependence $a(x, t, \lambda) = \lambda + a_1(x, t)$, we obtain nothing but the well-known Kaup–Boussinesq system

$$u_t^1 = u_x^2 - \frac{3}{2} u^1 u_x^1, \quad u_t^2 = \frac{1}{4} u_{xxx}^1 - u^2 u_x^1 - \frac{1}{2} u^1 u_x^2.$$

The General Problem. The Antonowicz–Fordy Construction

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between functions $u(x, t, \lambda)$ and $a(x, t, \lambda)$.

- Multi-component *rational* (with respect to the spectral parameter λ) generalization (ϵ_k are arbitrary parameters)

$$u(x, t, \lambda) = \frac{\lambda^M u^0(x, t) + \lambda^{M-1} u^1(x, t) + \dots + u^M(x, t)}{\epsilon_M \lambda^M + \epsilon_{M-1}\lambda^{M-1} + \dots + \epsilon_0}$$

The authors considered two main subclasses selected by the conditions: $\epsilon_M = 0$ and $u^0 = 1$ (the so called "Generalized KdV type systems"); $\epsilon_M = 0$ but $u^1 = 1$ (the so called "Generalized Harry Dym type systems"). In another paper written together with M. Marvan we found a third narrow subclass determined by a sole restriction $u^M = 0$.

- Now we consider ($M = 1, 2, ...$)

$$u(x, \mathbf{t}, \lambda) = \lambda^M \left(1 + \frac{u^1(x, \mathbf{t})}{\lambda} + \frac{u^2(x, \mathbf{t})}{\lambda^2} + \frac{u^3(x, \mathbf{t})}{\lambda^3} + ...\right),$$

where u^k are infinitely many unknown functions.

Integrable Dispersive Chains

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  where u^k are infinitely many unknown functions.
- The substitution and the linear dependence $a^{(1)} = \lambda + a_1(x, \mathbf{t})$ into

$$u_t = \left(-\frac{1}{2}\partial_x^3 + 2u\partial_x + u_x\right) a$$

  yields Mth integrable dispersive chain

$$u_t^k = u_x^{k+1} - \frac{1}{2}u^1 u_x^k - u^k u_x^1 + \frac{1}{4}\delta_M^k u_{xxx}^1, \quad k = 1, 2, \ldots,$$

  where δ_M^k is the Kronecker delta and

$$a_1 = -\frac{1}{2}u^1.$$

Higher Commuting Flows

- Higher commuting flows of the Korteweg de Vries hierarchy are determined by the linear spectral system

$$\psi_{xx} = (\lambda + u^1)\psi, \quad \psi_{t^k} = a^{(k)}\psi_x - \frac{1}{2}a_x^{(k)}\psi,$$

where

$$a^{(k)} = \lambda^k + \sum_{m=1}^{k} a_m \lambda^{k-m},$$

and functions a_m and u^1 depend on the "space" variable x and infinitely many extra "time" variables t^k (obviously, $t \equiv t^1$).

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- Substitution

$$u(x, \mathbf{t}, \lambda) = \lambda^M \left(1 + \frac{u^1(x,\mathbf{t})}{\lambda} + \frac{u^2(x,\mathbf{t})}{\lambda^2} + \frac{u^3(x,\mathbf{t})}{\lambda^3} + ...\right),$$

into

$$u_t = \left(-\frac{1}{2}\partial_x^3 + 2u\partial_x + u_x\right) a$$

Higher Commuting Flows

leads to higher commuting flows (here we define $a_0 = 1$)

$$u^k_{t^s} = \sum_{m=0}^{s} \left(u^{k+m}\partial_x + \partial_x u^{k+m} - \frac{1}{2}\delta_M^{k+m}\partial_x^3 \right) a_{s-m}, \quad s = 1, 2, \ldots,$$

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where all coefficients a_m can be found iteratively from the linear system (here we define $u^0 = 1$ and $u^{-m} = 0$ for all $m = 1, 2, ...$)

$$\sum_{m=0}^{s} \left(u^{m-k}\partial_x + \partial_x u^{m-k} - \frac{1}{2}\delta_M^{m-k}\partial_x^3 \right) a_{s-m} = 0, \quad k = 0, 1, ..., s-1.$$

For instance,

$$a_1 = -\frac{1}{2}u^1, \quad a_2 = -\frac{1}{2}u^2 + \frac{3}{8}(u^1)^2 - \frac{1}{8}\delta_M^1 u^1_{xx}, \quad a_3 = -\frac{1}{2}u^3 + \frac{3}{4}u^1 u^2$$

$$-\frac{5}{16}(u^1)^3 + \frac{1}{32}\delta_M^1(10u^1 u^1_{xx} + 5(u^1_x)^2 - u^1_{xxxx} - 4u^2_{xx}) - \frac{1}{8}\delta_M^2 u^1_{xx}, \ldots$$

Higher Commuting Flows

Thus all higher commuting flows are written also in an evolution form. For instance, the first commuting flow to

$$u_t^k = u_x^{k+1} - \frac{1}{2}u^1 u_x^k - u^k u_x^1 + \frac{1}{4}\delta_M^k u_{xxx}^1, \quad k = 1, 2, ...,$$

is (here we identify $y \equiv t^2$)

$$u_y^k = u_x^{k+2} - \frac{1}{2}u^1 u_x^{k+1} + \left(-\frac{1}{2}u^2 + \frac{3}{8}(u^1)^2 - \frac{1}{8}\delta_M^1 u_{xx}^1\right) u_x^k - u^{k+1} u_x^1$$

$$+u^k \left(-u_x^2 + \frac{3}{2}u^1 u_x^1 - \frac{1}{4}\delta_M^1 u_{xxx}^1\right) + \frac{1}{4}\delta_M^{k+1} u_{xxx}^1$$

$$+\frac{1}{4}\delta_M^k \left(u_{xxx}^2 - \frac{3}{4}[(u^1)^2]_{xxx} + \frac{1}{4}\delta_M^1 u_{xxxxx}^1\right).$$

Local Hamiltonian Structures

- A hierarchy of integrable dispersive chains

$$u_t^k = u_x^{k+1} - \frac{1}{2}u^1 u_x^k - u^k u_x^1 + \frac{1}{4}\delta_M^k u_{xxx}^1, \quad k = 1, 2, ...,$$

possesses *infinitely* many local Hamiltonian structures:

$$u_{t^s}^k = \sum_{m=1}^{s+1}\left(u^{k+m-1}\partial_x + \partial_x u^{k+m-1} - \frac{1}{2}\delta_M^{k+m-1}\partial_x^3\right)\frac{\delta \mathbf{H}_{s+1}}{\delta u^m};$$

$$u_{t^s}^1 = -2\partial_x \frac{\delta \mathbf{H}_{s+2}}{\delta u^1},$$

$$u_{t^s}^k = \sum_{m=2}^{s+2}\left(u^{k+m-2}\partial_x + \partial_x u^{k+m-2} - \frac{1}{2}\delta_M^{k+m-2}\partial_x^3\right)\frac{\delta \mathbf{H}_{s+2}}{\delta u^m};$$

$$u_{t^s}^1 = \text{-}2\partial_x \frac{\delta \mathbf{H}_{s+3}}{\delta u^2}, u_{t^s}^2 = \text{-}2\partial_x \frac{\delta \mathbf{H}_{s+3}}{\delta u^1} \text{-} \left(u^1\partial_x + \partial_x u^1 \text{-} \frac{1}{2}\delta_M^1\partial_x^3\right)\frac{\delta \mathbf{H}_{s+3}}{\delta u^2},$$

$$u_{t^s}^k = \sum_{m=3}^{s+3}\left(u^{k+m-3}\partial_x + \partial_x u^{k+m-3} - \frac{1}{2}\delta_M^{k+m-3}\partial_x^3\right)\frac{\delta \mathbf{H}_{s+3}}{\delta u^m}.$$

Conservation Laws

- All higher local conservation laws can be found from the observation

$$a_m = \frac{\delta \mathbf{H}_{m+s}}{\delta u^s}, \;\; m = 0, 1, ...; \;\; s = 1, 2, ...$$

In such a case all Hamiltonians can be found from above variation derivatives, for instance

$$\mathbf{H}_1 = \int u^1 dx, \quad \mathbf{H}_2 = \int \left(u^2 - \frac{1}{4}(u^1)^2 \right) dx,$$

$$\mathbf{H}_3 = \int \left(u^3 - \frac{1}{2}u^1 u^2 + \frac{1}{8}(u^1)^3 + \frac{1}{16}\delta^1_M (u^1_x)^2 \right) dx,$$

$$\mathbf{H}_4 = \int \Bigg(u^4 - \frac{1}{2}u^1 u^3 - \frac{1}{4}(u^2)^2 + \frac{3}{8}(u^1)^2 u^2 - \frac{5}{64}(u^1)^4$$
$$+ \frac{1}{32}\delta^1_M \left(-5u^1 (u^1_x)^2 - \frac{1}{2}(u^1_{xx})^2 + 4u^1_x u^2_x \right) + \frac{1}{16}\delta^2_M (u^1_x)^2 \Bigg) dx, ...$$

Elementary Reductions

Obviously for any natural number $N \geqslant M$ the reduction $u^{N+1} = 0$ of Mth dispersive chain

$$u_t^k = u_x^{k+1} - \frac{1}{2} u^1 u_x^k - u^k u_x^1 + \frac{1}{4} \delta_M^k u_{xxx}^1, \quad k = 1, 2, ...,$$

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leads to N component integrable dispersive systems:

1. $N = M = 1$, the Korteweg de Vries equation;

2. $N = 2, M = 1$, the Ito system

$$u_t^1 = u_x^2 - \frac{3}{2} u^1 u_x^1 + \frac{1}{4} u_{xxx}^1, \quad u_t^2 = -\frac{1}{2} u^1 u_x^2 - u^2 u_x^1;$$

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3. $N > 2, M = 1$,

$$u_t^1 = u_x^2 - \frac{3}{2} u^1 u_x^1 + \frac{1}{4} u_{xxx}^1,$$

$$u_t^k = u_x^{k+1} - \frac{1}{2} u^1 u_x^k - u^k u_x^1, \quad k = 2, \ldots, N-1,$$

$$u_t^N = -\frac{1}{2} u^1 u_x^N - u^N u_x^1;$$

Elementary Reductions

4. $N = M > 1$, (if $N = M = 2$, this is the Kaup–Boussinesq equation)

$$u_t^k = u_x^{k+1} - \frac{1}{2} u^1 u_x^k - u^k u_x^1, \quad k = 1, 2, ..., N-1,$$

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$$u_t^N = -\frac{1}{2}u^1 u_x^N - u^N u_x^1 + \frac{1}{4}u_{xxx}^1;$$

5. $N = M + 1, M > 1$,

$$u_t^k = u_x^{k+1} - \frac{1}{2}u^1 u_x^k - u^k u_x^1, \; k = 1, 2, ..., N-2,$$

$$u_t^{N-1} = u_x^N - \frac{1}{2}u^1 u_x^{N-1} - u^{N-1} u_x^1 + \frac{1}{4}u_{xxx}^1,$$

$$u_t^N = -\frac{1}{2}u^1 u_x^N - u^N u_x^1.$$

6. $N > M + 1, M > 1,$

$$u_t^k = u_x^{k+1} - \frac{1}{2} u^1 u_x^k - u^k u_x^1, \; k = 1, ..., M,$$

$$u_t^M = u_x^{M+1} - \frac{1}{2} u^1 u_x^M - u^M u_x^1 + \frac{1}{4} u_{xxx}^1,$$

$$u_t^k = u_x^{k+1} - \frac{1}{2} u^1 u_x^k - u^k u_x^1, \quad k = M + 1, ..., N - 1,$$

$$u_t^N = -\frac{1}{2} u^1 u_x^N - u^N u_x^1.$$

Rational Constraints with Movable Singularities

- Now we consider more complicated N component reductions $(M=1,2,...,\ \ K=0,1,...)$

$$u(x,\mathbf{t},\lambda)=\frac{\lambda^{M+K}+\lambda^{M+K-1}v_{M+K-1}(x,\mathbf{t})+...+\lambda v_1(x,\mathbf{t})+v_0(x,\mathbf{t})}{\lambda^K+\lambda^{K-1}w_{K-1}(x,\mathbf{t})+...+\lambda w_1(x,\mathbf{t})+w_0(x,\mathbf{t})}$$

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$$u(x, \mathbf{t}, \lambda) = \frac{\lambda^{M+K} + \lambda^{M+K-1} v_{M+K-1}(x, \mathbf{t}) + ... + \lambda v_1(x, \mathbf{t}) + v_0(x, \mathbf{t})}{\lambda^K + \lambda^{K-1} w_{K-1}(x, \mathbf{t}) + ... + \lambda w_1(x, \mathbf{t}) + w_0(x, \mathbf{t})}$$
- Suppose for simplicity that all roots of these two polynomials are pairwise distinct. Then the substitution $a^{(1)} = \lambda + a_1(x, \mathbf{t})$ and
$$u(x, \mathbf{t}, \lambda) = \frac{\prod_{m=1}^{M+K} (\lambda - s^m(x, \mathbf{t}))}{\prod_{k=1}^{K} (\lambda - r^k(x, \mathbf{t}))}$$
into
$$u_t = \left(-\frac{1}{2}\partial_x^3 + 2u\partial_x + u_x\right) a$$
yields new multi-component integrable dispersive systems!

Rational Constraints with Movable Singularities

- These new multi-component integrable dispersive systems are

$$r_t^k = (r^k + a_1) r_x^k, \quad s_t^i = (s^i + a_1) s_x^i + \frac{1}{2} \frac{\prod_{k=1}^{K} (s^i - r^k)}{\prod_{m \neq i} (s^i - s^m)} a_{1,xxx},$$

where

$$a_1 = \frac{1}{2} \left(\sum_{m=1}^{M+K} s^m - \sum_{k=1}^{K} r^k \right).$$

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where

$$a_1 = \frac{1}{2} \left(\sum_{m=1}^{M+K} s^m - \sum_{k=1}^{K} r^k \right).$$

- For instance, the Kaup–Boussinesq system becomes

$$s_t^1 = \frac{1}{2}(3s^1 + s^2) s_x^1 + \frac{(s^1 + s^2)_{xxx}}{4(s^1 - s^2)}, s_t^2 = \frac{1}{2}(s^1 + 3s^2) s_x^2 - \frac{(s^1 + s^2)_{xxx}}{4(s^1 - s^2)};$$

the Ito system takes the form

$$s_t^1 = \frac{1}{2}(3s^1 + s^2) s_x^1 + \frac{s^1 (s^1 + s^2)_{xxx}}{4(s^1 - s^2)}, s_t^2 = \frac{1}{2}(s^1 + 3s^2) s_x^2 - \frac{s^2 (s^1 + s^2)_{xxx}}{4(s^1 - s^2)}.$$

Three Dimensional Linearly Degenerate Quasilinear Equations

- The three dimensional quasilinear system

$$a_{1,t} = a_{2,x}, \quad a_1 a_{2,x} + a_{1,y} = a_2 a_{1,x} + a_{2,t},$$

is determined by the compatibility conditions

$$p_t = [(\lambda + a_1)p]_x, \quad p_y = [(\lambda^2 + a_1\lambda + a_2)p]_x,$$

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is determined by the compatibility conditions

$$p_t = [(\lambda + a_1)p]_x, \quad p_y = [(\lambda^2 + a_1\lambda + a_2)p]_x,$$

where $p = 1/\varphi$. Here $\varphi = \psi\psi^+$, where ψ and ψ^+ are two linearly independent solutions of

$$\psi_{xx} = u\psi,$$

$$\psi_t = (\lambda + a_1)\psi_x - \frac{1}{2}a_{1,x}\psi, \quad \psi_y = (\lambda^2 + a_1\lambda + a_2)\psi_x - \frac{1}{2}(\lambda a_{1,x} + a_{2,x})\psi.$$

Dispersive Reductions

- **Statement**: *Three dimensional quasilinear system*

$$a_{1,t} = a_{2,x}, \quad a_1 a_{2,x} + a_{1,y} = a_2 a_{1,x} + a_{2,t}$$

possesses infinitely many finite component differential constraints $a_1(\mathbf{u}, \mathbf{u}_x, \ldots)$, $a_2(\mathbf{u}, \mathbf{u}_x, \ldots)$, *where field variables* u^k *are solutions of dispersive integrable systems determined by linear spectral problem*

$$\psi_{xx} = u\psi,$$

$$\psi_t = (\lambda + a_1)\psi_x - \frac{1}{2}a_{1,x}\psi, \quad \psi_y = (\lambda^2 + a_1\lambda + a_2)\psi_x - \frac{1}{2}(\lambda a_{1,x} + a_{2,x})\psi,$$

where

$$u(x, t, y, \lambda) = \frac{\prod\limits_{m=1}^{M+K}(\lambda - s^m(x, t, y))}{\prod\limits_{k=1}^{K}(\lambda - r^k(x, t, y))}.$$

Dispersive Reductions

Differential constraints

$$a_1 = \frac{1}{2}\left(\sum_{m=1}^{M+K} s^m - \sum_{k=1}^{K} r^k\right), a_2 = \frac{1}{4}\sum_{m=1}^{M+K}(s^m)^2 - \frac{1}{4}\sum_{k=1}^{K}(r^k)^2 + \frac{1}{2}a_1^2 + \frac{1}{4}\delta_M^1 a_{1,xx}$$

and integrable commuting flows

$$r_t^k = (r^k + a_1)r_x^k, \quad s_t^i = (s^i + a_1)s_x^i + \frac{1}{2}\frac{\prod\limits_{k=1}^{K}(s^i - r^k)}{\prod\limits_{m\neq i}(s^i - s^m)}a_{1,xxx},$$

$$r_y^k = (a_2 + a_1 r^k + (r^k)^2)r_x^k,$$

$$s_y^i = (a_2 + a_1 s^i + (s^i)^2)s_x^i + \frac{1}{2}\frac{\prod\limits_{k=1}^{K}(s^i - r^k)}{\prod\limits_{m\neq i}(s^i - s^m)}(s^i a_{1,xxx} + a_{2,xxx}).$$

Differential Constraints. The Korteweg de Vries equation

Differential constraints are

$$a_1 = -\frac{1}{2}u^1, \quad a_2 = \frac{3}{8}(u^1)^2 - \frac{1}{8}u^1_{xx}.$$

Substitution into the three dimensional quasilinear system

$$a_{1,t} = a_{2,x}, \quad a_1 a_{2,x} + a_{1,y} = a_2 a_{1,x} + a_{2,t}$$

leads to an identity, if $u^1(x,t,y)$ is an arbitrary solution of the Korteweg de Vries equation

$$u^1_t = \left(\frac{1}{4}u^1_{xx} - \frac{3}{4}(u^1)^2\right)_x$$

and into its first commuting flow

$$u^1_y = \left(\frac{5}{8}(u^1)^3 - \frac{5}{16}(u^1_x)^2 - \frac{5}{8}u^1 u^1_{xx} + \frac{1}{16}u^1_{xxxx}\right)_x.$$