On classifiction and applications of pencils of Lie algebras

Algebraic curves with symmetries, their Jacobians and integrable dynamical systems

In memory of Petro Holod

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2 Plan

- Definition, examples and motivations
- Classification results
- Applications

3 Introduction: Lie pencils - definition and examples

Definition

A *bi-Lie structure* is a triple $(\mathfrak{g}, [\,,], [\,,]')$, where $\mathfrak{g}$ is a vector space and $[\,,], [\,,]'$ are two Lie algebra structures on $\mathfrak{g}$ which are *compatible*, i.e. so that any their linear combination $[\,,]^\lambda := [\,,] + \lambda[\,,]'$ is a Lie algebra structure. The whole family $\{[\,,]^\lambda\}$ of Lie brackets is a *Lie pencil*.

Example

Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$, $A \in \mathfrak{g}$ be a fixed matrix. Put

$$[x,_A y] = xAy - yAx.$$

Then $(\mathfrak{g}, [\,,], [\,,_A])$ is a bi-Lie structure, ($[\,,]$ the standard commutator).

Example (Kantor–Persits, 1988)

Let $\mathfrak{g} = \mathfrak{so}(n, \mathbb{K})$, $A \in \mathrm{Symm}(n, \mathbb{K})$, a fixed symmetric matrix. Then $(\mathfrak{g}, [\,,], [\,,_A])$ is a bi-Lie structure.

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4 Introduction: Motivation I - classical R-matrix formalism

Quasigraded Lie algebras

A Lie algebra $(\tilde{\mathfrak{g}}, [,])$ with a decomposition $\tilde{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ is *quasigraded of degree 1* if $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \oplus \mathfrak{g}_{i+j+1}$

Quasigraded Lie algebras $\rightarrow$ standard classical R-matrix

One checks that $\mathfrak{g}_+ := \bigoplus_{n\geq 0} \mathfrak{g}_n, \mathfrak{g}_- := \bigoplus_{n<0} \mathfrak{g}_n$ are subalgebras.

Bi-Lie structures $\rightarrow$ quasigraded Lie algebras

Let $(\mathfrak{g}, [,]_0, [,]_1)$ be a bi-Lie structure, $\tilde{\mathfrak{g}} := \mathfrak{g}[\lambda, 1/\lambda]$. Put $[,] = [,]_0 + \lambda[,]_1$ and extend this bracket to $\tilde{\mathfrak{g}}$. Then $\tilde{\mathfrak{g}}$ is quasigraded of degree 1.

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5 Introduction: Motivation I - classical R-matrix formalism

Applications

- Landau-Livshits PDE (the $\mathfrak{so}(n, \mathbb{R})$ bi-Lie structure, $n = 3$, Holod 1987)
- Other finite- and infinite-dimensional systems (Skrypnyk, Golubchik–Sokolov, Yanovski)

Introduction: Motivation II - bihamiltonian structures

Definition

A *bihamiltonian* structure on a manifold M is a pair $\eta, \eta' \in \Gamma(\bigwedge^2 TM)$ such that any linear combination $\eta^\lambda := \eta + \lambda\eta'$ is Poisson. We say that $\{\eta^\lambda\}$ is a *Poisson pencil.*

Remark

There is a 1-1 correspondence:

Lie pencils on $\mathfrak{g}$ $\leftarrow\text{- - - -}\rightarrow$ *Poisson pencils of Lie–Poisson structures on* $\mathfrak{g}^*$

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Introduction: Motivation II - bihamiltonian structures

Applications

Applications of the $\mathfrak{so}(n, \mathbb{R})$ bi-Lie structure:

- Manakov top (n-dimensional free rigid body), here A is diagonal, the "inertia tensor" of the body (Bolsinov 1992)
- Klebsh–Perelomov case (Bolsinov 1992)

Another bi-Lie structure on $\mathfrak{so}(n, \mathbb{R}) \times \mathfrak{so}(n, \mathbb{R})$

- Generalized Steklov–Lyapunov systems (Bolsinov–Fedorov 1992)

Useful notation

Let $\mathfrak{g}$ be a Lie algebra and $N : \mathfrak{g} \to \mathfrak{g}$ a linear operator. Put

$$[x, y]_N := [Nx, y] + [x, Ny] - N[x, y].$$

Definition

Let $\{[,]^v\}_{v \in V}$ be a n-dimensional vector space of Lie structures on a vector space $\mathfrak{g}$. It is called *irreducible* if the Lie algebras $(\mathfrak{g}, [,]^v)$ do not have common nontrivial ideals and *closed* if

$$\forall x \in \mathfrak{g}\ \forall v, w \in V\ \exists u \in V : [,]^v_{\mathrm{ad}^w x} := [,]^u, \mathrm{ad}^w x(y) = [x, y]^w.$$

8 Classification results: the Kantor–Persits theorem

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Kantor–Persits 1988 (announced only)

The list of irreducible closed vector spaces of Lie structures:

- $\mathfrak{g} = \mathfrak{so}(n, \mathbb{K}), \{[,_A]\}_{A \in \mathrm{Symm}(n,\mathbb{K})}$
- $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{K}), \{[,_A]\}_{A \in \mathfrak{m}(n,\mathbb{K})}$
- several *nonsemisimple* cases

here

$$[X,_A Y] := XAY - YAX,$$

$\mathfrak{sp}(n, \mathbb{K}) = \{X \in \mathfrak{gl}(2n, \mathbb{K}) \mid XJ + JX^T = 0\}$ the symplectic Lie algebra, $\mathfrak{m}(n, \mathbb{K}) := \{X \in \mathfrak{gl}(2n, \mathbb{K}) \mid XJ - JX^T = 0\}$ its orthogonal complement in $\mathfrak{gl}(2n, \mathbb{K})$ w.r.t. "trace form"

10 Classification results: the Odesskii–Sokolov theorem

Odesskii–Sokolov 2006

Classification of "bi-associative structures" $(\cdot, \circ)$ on $\mathfrak{gl}(n, \mathbb{K})$ $\Longrightarrow$ Examples of bi-Lie structures on $\mathfrak{gl}(n, \mathbb{K})$ (which do not restrict to $\mathfrak{sl}(n, \mathbb{K})$)

11 More examples of Lie pencils

Definition

Say that a bi-Lie structure $\mathcal{B} := (\mathfrak{g}, [,], [,]')$ is *semisimple* if $(\mathfrak{g}, [,])$ is semisimple.

Known examples of semisimple bi-Lie structures

KP1 $(\mathfrak{so}(n, \mathbb{C}), [,], [,_A])$ (Kantor–Persits 1988)

KP2 $(\mathfrak{sp}(n, \mathbb{C}), [,], [,_A])$ (Kantor–Persits 1988)

GS1 Let $(\mathfrak{g}, [,])$ be semisimple. There exists a bi-Lie structure related to any $\mathbb{Z}_n$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ on $(\mathfrak{g}, [,])$ and a decomposition of the subalgebra $\mathfrak{g}_0 = \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2$ to two subalgebras (Golubchik–Sokolov 2002)

P Let $(\mathfrak{g}, [,])$ be semisimple. There exists a bi-Lie structure related to any parabolic subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ (P 2006)

GS2 Examples on $\mathfrak{sl}(3, \mathbb{C}), \mathfrak{so}(4, \mathbb{C})$ related to $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings (Golubchik–Sokolov 2002)

12 Semisimple bi-Lie structures and operators

Obvious or Easy:

Let $(\mathfrak{g}, [,])$ *be a Lie algebra,* $[,]'$ *a bilinear bracket.*

- $[,]'$ *"compatible" with* $[,] \iff [,]'$ *is a 2-cocycle on* $(\mathfrak{g}, [,])$
- *In particular, if* $(\mathfrak{g}, [,], [,]')$ *is a semisimple bi-Lie str., then* $[,]' = [,]_W = [W\cdot, \cdot] + [\cdot, W\cdot] - W[\cdot, \cdot]$ *for some* $W : \mathfrak{g} \to \mathfrak{g}$
- *(Magri–Kosmann-Schwarzbach)* $[,]_N$ *is a Lie bracket for some* $N : \mathfrak{g} \to \mathfrak{g} \iff T_N(\cdot, \cdot) := [N\cdot, N\cdot] - N[\cdot, \cdot]_N$ *is a 2-cocycle on* $(\mathfrak{g}, [,])$
- *In particular,* $(\mathfrak{g}, [,], [,]')$ *is a semisimple bi-Lie str.* $\iff [,]' = [,]_W$ *and*

$$T_W(\cdot, \cdot) = [\cdot, \cdot]_P,$$

where $P : \mathfrak{g} \to \mathfrak{g}$ *is another linear operator. Moreover, the operators* W, P *are defined up to adding of inner differentiations* $\mathrm{ad}\, x$*.*

$$T_N(X, Y) = [NX, NY] - N([NX, Y] + [X, NY] - N[X, Y])$$

13 Semisimple bi-Lie structures and operators

Definition

Given a semisimple bi-Lie structure $\mathcal{B}$ call W such that $[,]' = [,]_W$ a *leading operator* for $\mathcal{B}$ and P a *primitive* for W. They satisfy *the main identity (MI)*

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Definition

Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there exists a direct decomposition $\mathrm{End}(\mathfrak{g}) = \mathrm{ad}\,\mathfrak{g} \oplus C$, where $C = (\mathrm{ad}\,\mathfrak{g})^{\perp}$ is the direct complement to $\mathrm{ad}\,\mathfrak{g} \subset \mathrm{End}(\mathfrak{g})$ w.r.t. the trace form. An operator $W \in \mathrm{End}(\mathfrak{g})$ is called *principal* if $W \in C$.

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14 Semisimple bi-Lie structures and operators

Theorem

1. *There exists a unique principal operator W with the property $[,]' = [,]_W$. Call it the* principal *(leading) operator of a bi-Lie structure $(\mathfrak{g}, [,], [,]')$.*
2. *If W is the principal operator, there exists a unique operator P primitive for W which is* symmetric *w.r.t. the Killing form on $\mathfrak{g}$.*

Example

For $\mathfrak{so}(n, \mathbb{K})$ bi-Lie structure we have $W = (1/2)(L_A + R_A)$ (operators of left and right multiplication by A).

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We say that bi-Lie structures $(\mathfrak{g}, [,], [,]')$ and $(\mathfrak{g}, [,], [,]'')$ are *strongly isomorphic* (*isomorphic*) if there exists an automorphism of the Lie algebra $(\mathfrak{g}, [,])$ sending the bracket $[,]'$ to $[,]''$ (to a linear combination $\alpha_1[,] + \alpha_2[,]''$).

Theorem

(P 2013) Let $(\mathfrak{g}, [,], [,]')$ and $(\mathfrak{g}, [,], [,]'')$ be two semisimple bi-Lie structures and let W', W'' be the corresponding principal operators. Then the bi-Lie structures are strongly isomorphic if and only if there exists an automorphism ϕ of the Lie algebra $(\mathfrak{g}, [,])$ with the property $\phi \circ W' = W'' \circ \phi$.

In particular, classification of semisimple bi-Lie structures up to isomorphism $\Longleftrightarrow$ classification of principal operators satisfyting MI up to action of automorphisms, rescaling, and adding scalar operators

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Classification results: continuation

Theorem

(P 2013) Consider a family of semisimple bi-Lie structures $(\mathfrak{g}, [\,,], [\,,]_W)$ *such that the principal leading operator* W *preserves the root decomposition*

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

with respect to some Cartan subalgebra $\mathfrak{h}$ *and is diagonal on* $\mathfrak{h}$*. Then this family splits to two disjoint classes of bi-Lie structures which are characterized as follows:*

- *Class I: The operator* W *preserves a* $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$*-grading* $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ *on* $(\mathfrak{g}, [\,,])$ *which is a coarsening of the root grading. The restriction* $W|\bigoplus_{i \neq 0} \mathfrak{g}_i$ *is symmetric.*
- *Class II: (*a conjecture proven for $\mathfrak{sl}(n, \mathbb{C})$*) The operator* W *preserves a* $\mathbb{Z}_n$*-grading,* $n > 2$*,* $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ *on* $(\mathfrak{g}, [\,,])$ *which is a coarsening of the root grading. The restriction* $W|\bigoplus_{i \neq 0} \mathfrak{g}_i$ *has a nontrivial antisymmetric part.*

Class II: (P)$\subset$(GS1')=$\{$bi-Lie structures related to $\mathbb{Z}_n$-gradings defined by an *inner* automorphism of n-th order$\}$.

Class I:

1. (KP1'):$(\mathfrak{so}(k,\mathbb{C}),[\,,],[\,_A]), A = \mathrm{diag}(t_1, t_1, t_2, t_2, \ldots, t_n, t_n, (t_{n+1}))$
2. (KP2):$(\mathfrak{sp}(n,\mathbb{C}),[\,,],[\,_A]), A = \mathrm{diag}(t_1, t_1, t_2, t_2, \ldots, t_n, t_n)$
3. (New examles)
   $(\mathfrak{sl}(n,\mathbb{C}),[\,,],[\,,]_W), WX := \frac{1}{2}((AX + XA) - Tr(AX + XA)B), A = \mathrm{diag}(t_1, \ldots, t_n), B = \mathrm{diag}(0, \ldots, 0, 1)$

Remark: In examples (GS2) the corresponding operator W do not preserve the root decomposition.

18 Applications: functions in involution

General principle I

Given a Poisson pencil $\{\eta^\lambda\}$, (roughly speaking) the Casimir functions of η^λ are in involution w.r.t. any Poisson bracket from the pencil.

Applications: functions in involution

An application of general principle I to Lie(–Poisson) pencils:

- Assume $\mathfrak{g}^0 := (\mathfrak{g}, [\,,\,])$ is semisimple. Then we can identify $\mathfrak{g} \cong \mathfrak{g}^*$ by means of the Killing form. The space of the Casimir functions is generated by the space of the invariant polynomials $\mathrm{Inv}(\mathfrak{g}^0)$.
- Usually one can find a linear map: $U_\lambda : \mathfrak{g} \to \mathfrak{g}$ such that
  - $\forall_{X,Y}\ [U_\lambda X, U_\lambda Y] = U_\lambda [X,Y]^\lambda$ (a homomorphism property).
  - $U_\lambda : \mathfrak{g} \to \mathfrak{g}$ is nondegenerate for almost all λ.
- Then the functions from the following set are in involution w.r.t. $\mathfrak{g}^0$

$$\mathcal{F} = \bigcup_{a.a.\lambda} \{F((U_\lambda^{-1})^* X) \mid F \in \mathrm{Inv}(\mathfrak{g}^0)\}.$$

Example: $\mathfrak{g} = \mathfrak{so}(n, \mathbb{K}), [\,,\,]^\lambda = [\,,\,] + \lambda[\,,_A\,], A = \mathrm{diag}(a_1, \ldots, a_n)$

$\mathrm{Inv}(\mathfrak{so}(2k, \mathbb{K})) = \mathrm{Fun}(\mathrm{Tr}(X^2), \ldots, \mathrm{Tr}(X^{2(k-1)}), \mathrm{Pf}(X))$,
$\mathrm{Inv}(\mathfrak{so}(2k+1, \mathbb{K})) = \mathrm{Fun}(\mathrm{Tr}(X^2), \ldots, \mathrm{Tr}(X^{2k}))$,
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Applications: completeness of functions in involution

General principle II

If for a generic $x \in M$ we have $\operatorname{rank} \eta^\lambda_{\mathbb{C},x} = const$ over $\lambda \in \mathbb{C}$, where $\eta^\lambda_{\mathbb{C}}$ is the complexification of η^λ (*Kronecker* case), then the set $\mathcal{F}$ is a *complete* family of functions in involution (w.r.t. any bracket of the pencil).

An application of the general principle II to Lie(–Poisson) pencils:

Theorem (Bolsinov, 1992)

Let $\{\mathfrak{g}^\lambda\}, \mathfrak{g}^\lambda = (\mathfrak{g}, [\,,]^\lambda)$, be a Lie pencil over $\mathbb{R}$, where $[\,,]^\lambda = [\,,] + \lambda[\,,]$ and $\mathfrak{g}^0 = (\mathfrak{g}, [\,,])$ is semisimple. Assume

1. there exists an isomorphism $U_\lambda : \mathfrak{g}^\lambda_{\mathbb{C}} \to \mathfrak{g}^0_{\mathbb{C}}$ for all $\lambda \in \mathbb{C} \setminus E$, where E is a finite set; here $\mathfrak{g}^\lambda_{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{g}^\lambda$.
2. $\operatorname{ind} \mathfrak{g}^\lambda_{\mathbb{C}} = \operatorname{ind}(\mathfrak{g}^0)$ for all $\lambda \in E$ (index$=$ codim of generic coadjoint orbit).

Then the set of functions in involution
$\mathcal{F} = \bigcup_{\lambda \in \mathbb{R} \setminus E} \{F((U_\lambda^{-1})^* X) \mid F \in \operatorname{Inv}(\mathfrak{g}^0)\}$ is *complete*.

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21 Applications: Manakov top

Example (Bolsinov 1992)

Let $\mathfrak{g} = \mathfrak{so}(n, \mathbb{K}), [\,,]^\lambda = [\,,] + \lambda[\,,_A], A = \mathrm{diag}(a_1, \ldots, a_n), a_i > 0$. Then

$$E = \{-1/a_1, \ldots, -1/a_n\}, \mathfrak{g}^{-1/a_i} \cong \mathfrak{e}(n-1) = \mathfrak{so}(n-1, \mathbb{K}) \ltimes \mathbb{K}^{n-1}$$

and $\mathrm{ind}\,\mathfrak{e}(n-1, \mathbb{K}) = \mathrm{ind}\,\mathfrak{so}(n, \mathbb{K})$. The Manakov hamiltonian $\mathrm{Tr}(MX \cdot X)$ is among the functions in involution constructed above; here $M(E_{ij} - E_{ji}) = (1/(\sqrt{a_i} + \sqrt{a_j}))(E_{ij} - E_{ji})$.

Theorem (Haine 1984)

The only operators $M : \mathfrak{so}(n, \mathbb{C}) \to \mathfrak{so}(n, \mathbb{C})$, which are diagonal in the basis $\{E_{ij} - E_{ji}\}$, such that the corresponding Euler equations $\dot{X} = [X, MX]$ have single valued solutions and are integrable in quadratures are of the "generalized Manakov type":

$$M(E_{ij} - E_{ji}) = \frac{b_i - b_j}{a_i - a_j}(E_{ij} - E_{ji}).$$

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22 Applications: geodesic flows

Question

Which quadratic hamiltonians are among the functions in involution constructed by a Lie pencil?

Answer

Since $B(X,X) \in \mathrm{Inv}(\mathfrak{g}^0)$, where $B(X,Y)$ is the Killing form on $\mathfrak{g}^0$, the family of quadratic hamiltonians is $\mathcal{Q} = \bigcup_{\lambda \notin E} \{B((U_\lambda^{-1})^* X, (U_\lambda^{-1})^* X)\} = = \bigcup_{\lambda \notin E} \{B((U_\lambda^* U_\lambda)^{-1} X, X)\}$.

Remark 1

If $U_\lambda \in \mathrm{End}(\mathfrak{g})$ satisfies $[U_\lambda \cdot, U_\lambda \cdot] = U_\lambda [\cdot, \cdot]^\lambda$, where $[\,,]^\lambda = [\,,] + \lambda[\,,]$, then $U'_\lambda = \phi \circ U_\lambda \circ \psi_\lambda$ also satisfies this identity; here $\phi \in \mathrm{Aut}(\mathfrak{g}^0)$, $\psi_\lambda \in \mathrm{Aut}(\mathfrak{g}, [\,,]^\lambda)$. However, $(U'_\lambda)^* U'_\lambda = U_\lambda^* U_\lambda$ and $(U_\lambda^* U_\lambda)^{-1}$ is defined invariantly.

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23 Applications: geodesic flows

Remark 2

$(U_\lambda^* U_\lambda)^{-1} = I - \lambda(W + W^*) + o(\lambda)$ In particular, if the hypotheses of the Bolsinov theorem are satisfied, the operator $W + W^*$ gives a metric with an integrable geodesic flow.

Theorem

Let $\{\mathfrak{g}^\lambda\}, \mathfrak{g}^\lambda = (\mathfrak{su}(n), [\,,]^\lambda)$, be a Lie pencil defined by the bi-Lie structure $(\mathfrak{su}(n), [\,,], [\,,]_W)$, $WX := \frac{1}{2}((AX + XA) - Tr(AX + XA)B)$, $A = \mathrm{diag}(t_1, \ldots, t_n)$, $B = \mathrm{diag}(0, \ldots, 0, 1)$. Then it satisfies the hypotheses of the Bolsinov theorem.

23 Applications: geodesic flows

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Integrable geodesic flow on $\mathfrak{su}(4)$

Corollary 1a

On $\mathfrak{su}(4)$ we get a 4-parameter family of metrics $\mathrm{Tr}((MX)Y)$ with completely integrable geodesic flows, where

- $Mx_{ij} = (a_i + a_j)x_{ij}$ for $i \neq j$,
- $$M\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} a_2 + a_3 & a_3 - \frac{1}{2}(a_1 + a_2) & a_2 - \frac{1}{2}(a_1 + a_3) \\ a_3 - \frac{1}{2}(a_1 + a_2) & a_1 + a_3 & a_1 - \frac{1}{2}(a_2 + a_3) \\ a_2 - \frac{1}{2}(a_1 + a_3) & a_1 - \frac{1}{2}(a_2 + a_3) & a_1 + a_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

Here $\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -x_{11} + x_{22} + x_{33} - x_{44} \\ x_{11} - x_{22} + x_{33} - x_{44} \\ x_{11} + x_{22} - x_{33} - x_{44} \end{bmatrix}$ is an orthonormal basis in $\mathfrak{h}$.

25 Applications: integrable geodesic flow on $\mathfrak{so}(6,\mathbb{R})$

Corollary 1b

In view of $\mathfrak{su}(4) \cong \mathfrak{so}(6,\mathbb{R})$ we get on $\mathfrak{so}(6,\mathbb{R})$ a 4-parameter family of metrics $\mathrm{Tr}((MX)Y)$ with completely integrable geodesic flows, which are not diagonal in the $E_{ij} - E_{ji}$ basis. Explicitly:

$$\mathfrak{so}(6,\mathbb{R}) \ni X = \left(\begin{array}{cc|cc|cc}
0 & X^{11} & X^{12}_{11} & X^{12}_{12} & X^{13}_{11} & X^{13}_{12} \\
 & 0 & X^{12}_{21} & X^{12}_{22} & X^{13}_{21} & X^{13}_{22} \\
\hline
 & & 0 & X^{22} & X^{23}_{11} & X^{23}_{12} \\
 & & & 0 & X^{23}_{21} & X^{23}_{22} \\
\hline
 & & & & 0 & X^{33} \\
 & & & & & 0
\end{array}\right)$$

$$MX^{12}_{11} = \tfrac{1}{2}((a_1 + a_2 + a_3 + a_4)X^{12}_{11} \quad +(a_1 + a_2 - a_3 - a_4)X^{12}_{22})$$
$$MX^{12}_{22} = \tfrac{1}{2}((a_1 + a_2 - a_3 - a_4)X^{12}_{11} \quad +(a_1 + a_2 + a_3 + a_4)X^{12}_{22})$$
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$$MX^{12}_{21} = \tfrac{1}{2}((-a_1 - a_2 + a_3 + a_4)X^{12}_{12} \quad +(a_1 + a_2 + a_3 + a_4)X^{12}_{21})$$

25 Applications: integrable geodesic flow on $\mathfrak{so}(6,\mathbb{R})$

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0	X^{11}	X^{12}_{11}	X^{12}_{12}	X^{13}_{11}	X^{13}_{12}
	0	X^{12}_{21}	X^{12}_{22}	X^{13}_{21}	X^{13}_{22}
		0	X^{22}	X^{23}_{11}	X^{23}_{12}
			0	X^{23}_{21}	X^{23}_{22}
				0	X^{33}
					0

$$MX^{12}_{11} = \tfrac{1}{2}((a_1 + a_2 + a_3 + a_4)X^{12}_{11} \qquad +(a_1 + a_2 - a_3 - a_4)X^{12}_{22})$$
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26 Applications: integrable geodesic flow on $\mathfrak{so}(6, \mathbb{R})$

$$M \begin{bmatrix} X^{11} \\ X^{22} \\ X^{33} \end{bmatrix} = \begin{bmatrix} a_2 + a_3 & a_3 - \frac{1}{2}(a_1 + a_2) & a_2 - \frac{1}{2}(a_1 + a_3) \\ a_3 - \frac{1}{2}(a_1 + a_2) & a_1 + a_3 & a_1 - \frac{1}{2}(a_2 + a_3) \\ a_2 - \frac{1}{2}(a_1 + a_3) & a_1 - \frac{1}{2}(a_2 + a_3) & a_1 + a_2 \end{bmatrix} \begin{bmatrix} X^{11} \\ X^{22} \\ X^{33} \end{bmatrix}$$

27 Perspectives

- Classification of Lie pencils.
  - To classify those operators W which do not preserve the root grading $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$ in complex case, but are scalar on some other gradings.
  - Real Lie agebras.
  - The same problem admitting nilpotent operators.
- Analysis of algebraic structure of known examples. When they are Kronecker? What, if they are not Kronecker?