An Application of the Theory of Tau Function to Riemann's Theta Functions

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The notion of tau function of an integrable system can be considered as an extension of the notion of theta function. Compared with theta functions, the tau function carries a universal structure which is common for all solutions of the integrable system. Therefore, in particular, the tau function is considered to know the common structure of theta functions of Riemann surfaces.

So it is quite natural to study theta functions by way of tau function. The most successful example of the study from this point of view is the solution of the Novikov's conjecture by Shiota.

In this talk I will add another example. It is the Riemann's singularity theorem. This classical theorem is a theorem describing the analytic properties of the theta function in terms of the geometry of a Riemann surface.

More precisely the aim of this talk is

(1) to determine the initial term of the expansion of the Riemann's theta function of an arbitrary Riemann surface at any point on the theta divisor,

(2) to formulate and prove the refinement of the Riemann's singularity theorem,

(3) to propose a normalization factor of the sigma function with characteristics so that the sigma function becomes modular invariant.

Let me begin by the review ofl the Riemann's singularity theorem.

X: a compact Riemann suface of genus g,

$\{\alpha_i, \beta_i\} \to \{dv_i\} \to \Omega = (\int_{\beta_j} dv_i) \to$

$\to \theta(z, \Omega), \quad J(X) = \mathbb{C}/(\mathbb{Z}^g + \Omega\mathbb{Z}).$

$p_\infty \in X \to \delta$: Riemann's constant, Abel-Jacobi map

$$I\colon X \to J(X),$$
$$I(p) = \int_{p_\infty}^{p} dv.$$

We identify a degree zero divisor with its Abel-Jacobi image:

$$\sum (p_i - q_i) = \sum (I(p_i) - I(q_i)).$$

I particular, for the Riemann divisor Δ

$$\Delta - (g-1)p_\infty = \delta.$$

For $p_1, ..., p_n \in X$ set

- $L(p_1 + \cdots + p_n) := \{$a meromorphic function on X whose pole divisor is bounded by $p_1 + \cdots + p_n \}$,

- $d(p_1 + \cdots + p_n) :=$ dim $L(p_1 + \cdots + p_n)$.

For $\alpha = (\alpha_1, ..., \alpha_g)$ set

- $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_g^{\alpha_g}, \quad ||\alpha|| = \alpha_1 + \cdots + \alpha_g$.

  We call $||\alpha||$ the degree of ∂^α.

Riemann's vanishing theorem

$$\theta(z) = 0$$

$$\Leftrightarrow$$

$$z = p_1 + \cdots + p_{g-1} - \Delta \text{ for some } p_i \in X.$$

Riemann's singularity theorem

Suppose that $e = p_1 + \cdots + p_{g-1} - \Delta$ and $r = d(p_1 + \cdots + p_{g-1})$. Then the multiplicity of e is r:

(1) $\partial^\alpha \theta(e) = 0$ for $||\alpha|| < r$.

(2) $\partial^\beta \theta(e) \neq 0$ for some β with $||\beta|| = r$.

In the study of the sigma functions, Onishi('05) found one of such β explicitly to each zero of the theta function in the case of hyperelliptic curves.

In this talk the Onishi's results are extended to the case of an arbitrary Riemann surface.

To clarify the situation, let me restart by the data:

$(X, p_\infty, \{\alpha_i, \beta_i\})$: as before

e: $\theta(e) = 0$.

Strategy of the arguments

$$(X, p_\infty, \{\alpha_i, \beta_i\}, e)$$

$$\Downarrow$$

Schur functions $\{s_\mu(t)\}_{\mu \geq \lambda}$ (combinatorics)

$$\Downarrow$$

Tau functions $\tau(t)$ (integrable systems)

$$\Downarrow$$

$$\theta(z)$$

Schur function

Define the polynomial $p_n(t)$, $t = (t_1, t_2, ...)$, by

$$\exp(\sum_{n=1}^{\infty} t_n k^n) = \sum_{n=0}^{\infty} p_n(t) k^n.$$

Example. $p_0 = 1$, $p_1 = t_1$, $p_2 = t_2 + t_1^2/2$,

$$p_3 = t_3 + t_1 t_2 + t_1^3/3!.$$

Schur function

The Schur function $s_\lambda(t)$ corresponding to a partition $\lambda = (\lambda_1, ..., \lambda_l)$ is defined by the Jacobi-Trudi formula:

$$s_\lambda(t) = \det(p_{\lambda_i - i + j}(t))_{1 \le i,j \le l}.$$

Example

$$s_{(4)}(t) = t_4 + t_3 t_1 + \frac{1}{2} t_2^2 + \frac{1}{2} t_2 t_1^2 + \frac{1}{24} t_1^4,$$

$$s_{(3,1)}(t) = -t_4 - \frac{1}{2} t_2^2 + \frac{1}{2} t_1^2 t_2 + \frac{1}{8} t_1^4,$$

$$s_{(2,2)}(t) = -t_3 t_1 + t_2^2 + \frac{1}{12} t_1^4,$$

$$s_{(2,1,1)}(t) = t_4 - \frac{1}{2} t_2^2 - \frac{1}{2} t_2 t_1^2 + \frac{1}{8} t_1^4$$

$$s_{(1,1,1,1)}(t) = -t_4 + t_3 t_1 + \frac{1}{2} t_2^2 - \frac{1}{2} t_2 t_1^2 + \frac{1}{24} t_1^4, \ .$$

We define weight and degree by

$$\mathrm{wt}\, t_i = i$$

$$\mathrm{wt}\, \lambda = |\lambda| = \lambda_1 + \cdots + \lambda_l,$$

$$\deg t_i = 1.$$

Then the Schur function $s_\lambda(t)$ is homogeneous with the weight $|\lambda|$. It is inhomogeneous with respect to degree in general.

We will assign a partition to each geometric data of Riemann surfaces.

We write the zero e of the theta function as

$$e = q_1 + \cdots + q_{g-1} - \Delta.$$

Let L be the flat line bundle on X defined by

$$L = q_1 + \cdots + q_{g-1} - (g-1)p_\infty$$
$$= e + \delta.$$

Gaps and non-gaps

A non-negative integer b is called a gap of L at p_∞ if there does not exist a meromorphic section of L which has a pole only at p_∞ of order b.

By Riemann-Roch we can easily prove that there are exactly g gaps for any (L, p_∞).

An element of the complement of gaps in the set of non-negative integers is called a non-gap.

We consider two kinds of gap sequences simultaneously.

$1 = w_1 < \cdots < w_g$: the gap sequence at p_∞,

$0 = w_1^* < w_2^* < \cdots$: the non-gaps at p_∞,

$0 \le b_1 < \cdots < b_g$: the gap sequence of L at p_∞,

$0 \le b_1^* < b_2^* < \cdots$: the non-gaps of L at p_∞.

Define

$$\lambda = (b_g, ..., b_1) - (g-1, ..., 1, 0)$$

and consider the Schur function corresponding to λ.

A special property of the Schur function $s_\lambda(t)$ is given by the following proposition.

Schur function of a Riemann surface

Proposition

$s_\lambda(t)$ is a polynomial of the variables $t_{w_1},\ldots,t_{w_g}$ corresponding to the gap sequence $w_1, \ldots w_g$ of X at p_∞.

In order to connect the Schur function $s_\lambda(t)$ to the theta function we need to change the variables. It is described by a non-normalized period matrix.

We further specify

z: a local coordinate at p_∞,

$\{du_{w_i}\}$: a basis of holomorphic 1-forms such that

$$du_{w_i} = z^{w_i-1}(1+O(z))dz \quad \text{at} \quad p_\infty.$$

The period matrices are introduced by

$$2\omega_1 = \left(\int_{\alpha_j} du_{w_i}\right),\ 2\omega_2 = \left(\int_{\beta_j} du_{w_i}\right),\ \Omega = \omega_1^{-1}\omega_2.$$

We consider the function of the form

$$\theta((2\omega_1)^{-1}u + e),$$

where

$$u = {}^t(u_{w_1}, ..., u_{w_g}).$$

This enumeration of variables is for the sake of being consistent with the theory of the tau function of the KP-hierarchy.

The relation of the Schur function $s_\lambda(t)$ with the theta function is given by the following theorem.

The first term of the exapansion with respect to weights

Theorem

$$C\theta((2\omega_1)^{-1}u+e)=s_\lambda(t)|_{t_{w_i}=u_{w_i}}+\text{higher weight terms.}$$

Recall that $s_\lambda(t)$ is a polynomial of $t_{w_1},...,t_{w_g}$. So this formula is consistent with the enumeration of the variables $u={}^t(u_{w_1},...,u_{w_g})$ in the theta function.

Remarks

This type of expansion formula was previously known only for (X, e, p_∞) such that $e = -\delta$, $p_\infty =$ some special point and X being

- (n, s) curves (Buchstaber-Enolski-Leykin (1999), N (2010))
- telescopic curves (Ayano-N (2013))
- a certain space curve of genus two (Matsutani (2011))

The theorem is proved by using the Sato's theory of the KP-hierarchy.

This theory makes a one to one correspondence between solutions of the KP-hierarchy and points of a certain infinite dimensional Grassmanian called the universal Grassman manifold (UGM).

Strategy of the proof

$$\tau(t) = e^{\frac{1}{2}q(t)}\theta(At + e) : \text{ a solution to KP}$$

$$\Rightarrow U \in UGM^{\lambda}$$

$$\Rightarrow C\tau(t) = s_{\lambda}(t) + \sum_{\lambda<\mu} \xi_{\lambda\mu} s_{\mu}(t)$$

$\Rightarrow$ the expansion of theta.

X: any , p_∞: a non-Weierstrass point, $e = -\delta$

Then

$L = \mathcal{O}$ =the trivial line bundle,

$(w_1, ..., w_g) = (b_1, ..., b_g) = (1, 2, ..., g)$,

$\lambda = (b_g, ..., b_1) - (g-1, ..., 1) = (1, ..., 1)$.

$s_\lambda(t) = (-1)^g p_g(-t) = (-1)^g(-t_g + t_1 t_{g-1} + \cdots)$.

Example of the initial term -2-

X: any , p_∞: a non-Weierstrass point, $e = \delta$

Then

$$L = K_X((-2g-2)p_\infty)$$

$$(b_1, ..., b_g) = (0, 1, ..., g-2, 2g-1),$$

$$\lambda = (b_g, ..., b_1) - (g-1, ..., 1) = (g).$$

$$s_\lambda(t) = p_g(t) = t_g + t_1 t_{g-1} + \cdots.$$

Note that the partitions $(1,...,1)$ is the conjugate of the partition (g).

Actually this is a general structure and the following duality theorem is valid.

Duality

Theorem

We have

$$C\theta((2\omega_1)^{-1}u + e) = s_\lambda(t)|_{t_{w_i}=u_{w_i}} + \cdots,$$

$$C'\theta((2\omega_1)^{-1}u - e) = s_\mu(t)|_{t_{w_i}=u_{w_i}} + \cdots,$$

where μ is the conjugate partition of λ.

A family of Schur functions

In the expansion of the tau function the Schur functions $s_\mu(t)$ satisfying the condition $\mu \geq \lambda$ appear.

So we study the common properties of those Schur functions.

Let us define

$$m_0 = d(q_1 + \cdots + q_{g-1}) < g.$$

We define the "a-sequence" by

$$(a_1, \ldots, a_{m_0})$$

$$= (b_g, b_{g-1}, \ldots, b_{g-m_0+1}) - (b_1^*, \ldots, b_{m_0}^*).$$

Example of a-sequence

hyperelliptic curves, $p_\infty = \infty$, $q_i = p_\infty$ for any i:

$g = 2$: $m_0 = 1$, $a_1 = 3$,

$g = 3$: $m_0 = 2$, $(a_1, a_2) = (5, 1)$,

$g = 4$: $m_0 = 2$, $(a_1, a_2) = (7, 3)$,

$g = 5$: $m_0 = 3$, $(a_1, a_2, a_3) = (9, 5, 1)$,

$g = 6$: $m_0 = 3$, $(a_1, a_2, a_3) = (11, 7, 3)$.

Onishi's results (hyperelliptic) : $e = -\delta$ case

$$g = 2, \quad m_0 = 1, \quad \sigma_3(0) \neq 0,$$

$$g = 3, \quad m_0 = 2, \quad \sigma_{51}(0) \neq 0,$$

$$g = 4, \quad m_0 = 2, \quad \sigma_{73}(0) \neq 0,$$

$$g = 5, \quad m_0 = 3, \quad \sigma_{951}(0) \neq 0,$$

$$g = 6, \quad m_0 = 3, \quad \sigma_{11,7,3}(0) \neq 0,$$

$$\vdots \qquad \vdots$$

Thus the "a-sequence" is a natural generalization of Onishi's sequence.

Properties of the a-sequence

In general the *a*-sequence has the following properties.

Proposition

(1) (distinct) $a_1 > \cdots > a_{m_0} > 0$.

(2) (gaps) $a_i \in \{w_j\}$.

(3) (weight) $\sum a_i = |\lambda|$.

Vansihing and non-vanishing theorem for Schur functions

Let $\partial_i = \partial_{t_i}$.

Theorem

Let $\mu \geq \lambda$. Then

(1) $\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots s_\mu(0) = 0$ *if* $\sum i\alpha_i \neq |\mu|$. *(weight)*

(2) $\partial_{i_1}\cdots\partial_{i_m} s_\mu(0) = 0$ *if* $m < m_0$. *(degree)*

(3) $\partial_{a_1}\cdots\partial_{a_{m_0}} s_\lambda(0) \neq 0$.

These properties of Schur functions are lifted to the tau function of the KP hierarchy.

Using the expansion of the tau function in terms of Schur functions

$$\tau(t) = s_\lambda(t) + \sum_{\lambda<\mu} \xi_{\lambda\mu} s_\mu(t).$$

and the vanishing-non-vanishing theorem for Schur functions we can prove the following theorem for tau functions.

Vansihing and Non-vanishing theorem for tau functions

Theorem

(1) $\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\tau(0)=0$ *if* $\sum i\alpha_i<|\lambda|$. *(weight)*

(2) $\partial_{i_1}\cdots\partial_{i_m}\tau(0)=0$ *if* $m<m_0$. *(degree)*

(3) $\partial_{a_1}\cdots\partial_{a_{m_0}}\tau(0)\neq 0$.

Riemann's theta functions inherit those properties from tau functions.

Recall that our tau function has the expression in terms of the theta function of the form

$$\tau(t) = Ce^{\frac{1}{2}q(t)}\theta((2\omega_1)^{-1}(Bt) + e).$$

Here $B = (b_{ij})$ is the $g \times \infty$ matrix determined by the expansion of du_{w_i} and has the "triangular structure":

$$b_{ij} = \begin{cases} 0, & j < w_i \\ 1, & j = w_i \end{cases}$$

Using this triangular structure of B and the previous properties of the tau function we can prove the following theorem for theta functions.

Vansihing and non-vanishing theorem for theta functions

Theorem

(1) $\partial_{u_{w_1}}^{\alpha_1} \cdots \partial_{u_{w_g}}^{\alpha_g} \theta(e) = 0 \quad$ *if* $\quad \sum w_i \alpha_i < |\lambda|$. *(weight)*

(2) $\partial_{u_{i_1}} \cdots \partial_{u_{i_m}} \theta(e) = 0 \quad$ *if* $\quad m < m_0$. *(degree)*

(3) $\partial_{u_{a_1}} \cdots \partial_{a_{m_0}} \theta(e) \neq 0$.

The properties (2) and (3) give the Riemann's singularity theorem.

Moreover (3) gives one non-vanishing derivatives explicitly.

It is natural to ask whether it is possible to determine all the non-vanishing derivatives of degree m_0 at e.

This problem is not solved yet. Here I will comment partial results.

Minimal degree term of Schur function

It is possible to determine the minimal degree term of the Schur function $s_\lambda(t)$ corresponding to the zero e of the theta function.

Let $L_\lambda(t)$ be the minimal degree term of $s_\lambda(t)$:

$$s_\lambda(t) = L_\lambda(t) + \text{ higher degree terms.}$$

Formula for $L_\lambda((t)$

Theorem

Let $\lambda = (\lambda_1, ..., \lambda_g)$ *be the partition corresponding to e. Then* $L_\lambda(t)$ *is given by the determinant of a* $m_0 \times m_0$*-matrix as*

$$L_\lambda(t) = \det(t_{\lambda_i - i + j})_{1 \le i \le m_0, j \ne g-b_1,...,g-b_{g-m_0}}.$$

In the case of a hyperelliptic curve of genus g and $e = -\delta$ $\lambda = (g, g-1, ..., 1)$. In this case $m_0 = [\frac{g+1}{2}]$ and

for $g = 2k$

$$L_\lambda(t) = \det\big(t_{2k+1-2i+2j}\big)_{1 \le i,j \le k}$$

for $g = 2k+1$

$$L_\lambda(t) = \det\big(t_{2k+1-2i+2j}\big)_{1 \le i,j \le k+1}.$$

Comparison with the minimal degree term of the sigma function

These are precisely the minimal degree terms in the expansion of the hyperelliptic sigma functions discovered by Buchstaber, Enolski and Leykin in 1997.

I checked that similar phenomena occur also for $(3, 4)$, $(3, 5)$ and $(3, 7)$ curves.

Conjecture on the minimal degree term of the theta function

Conjecture

Let λ be the partition corresponding to a zero e of the theta function. Then

$$C\theta((2\omega_1)^{-1}u + e) = L_\lambda(u) + \textit{higher degree terms}$$

Applications to sigma functions

As another application of the determination of the initial term of the series expansion of the theta function at an arbitrary point of the theta divisor, it is possible to determine a normalization constant of the sigma function with an arbitrary characteristics such that it is modular invariant.

The higher genus sigma function was introduced by F. Klein for heperelliptic curves. The sigma functions for more general plane algebraic curves called (n,s) curves were introduced and studied by Buchstaber, Enolski and Leykin. The extensions to certain space curves are given by Matsutani and Ayano. Recently Korotkin and Shramchenko defined sigma functions with half integer characteristics for any Riemann surfaces.

We consider the sigma function dfined by Korotkin-Shramchenko except the normalization constant and that we consider arbitrary real characteristics.

We consider any zero e of $\theta(z)$ and write

$$e = \Omega\alpha + \beta, \ \alpha, \beta \in \mathbb{R}^g.$$

Let λ be the corresponding partition and $(w_1, \ldots, w_g)$ the gap sequence at p_∞ as before.

We set $u = {}^t(u_{w_1}, \ldots, u_{w_g})$.

Then

Expansion of the sigma function with arbitrary characteristics

Theorem

If we define

$$\sigma\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(u) = \exp(\frac{1}{2}{}^t u\eta_1\omega_1^{-1}u)\frac{\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}((2\omega_1)^{-1}u)}{\partial_{a_1}\cdots\partial_{a_{m_0}}\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0)}$$

then

$$\sigma\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(u) = \pm s_\lambda(u) + \cdots$$

and it is modular invariant.

Remark

Another normalization constant is proposed by Korotkin and Shramchenko (2012) for the sigma function to be (conjecturally) modular invariant.

Summary

We have studied the properties of the Schur functions. By combining them with the properties of the tau functions of the KP-hierarchy we have, for arbitrary Riemann surfaces,

(1) the initilal term of the series expansion of the theta function with respect to weight at any point on the theta divisor,

(2) a refined version of the Riemann's singularity theorem,

(3) a proper normalization constantnt of the sigma function with arbitrary characteristics.