Bergman tau-function and Witten classes

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References

- M.Bertola, D.Korotkin, "Bergman tau-function, Witten cycles and tautological classes on moduli spaces", in progress
- D.Korotkin and P.Zograf, *"Tau-function and moduli of differentials"*, Math. Res. Lett., **18** 447-458 (2011)
- A.Kokotov, D.Korotkin, P.Zograf, *"Isomonodromic tau function on the space of admissible covers"*, Adv. in Math. **227** 586-600 (2011)
- D.Korotkin, P.Zograf, *"Prym classes and tau-functions"*, Contemporary Math. (2013)

Basic notations

- C - Riemann surface of genus g; (a_α, b_α) - canonical basis in $H_1(C, \mathbb{Z})$, $\alpha = 1, \dots, g$; v_α - normalized basis of holomorphic 1-forms: $\oint_{a_\alpha} v_\beta = \delta_{\alpha\beta}$.
- Canonical bimeromorphic differentials

$$B(x,y) = d_x d_y \log E(x,y); \qquad \oint_{a_\alpha} B(\cdot, y) = 0$$

- Bergman projective connection: as $y \to x$,

$$B(x,y) = \left(\frac{1}{(\xi(x) - \xi(y))^2} + \frac{1}{6} S_B(\xi(x)) + \dots \right) d\xi(x) d\xi(y)$$

- CFT with $c = 1$ (free bosons): (zz)-component of EM tensor: $T_{zz} = \frac{1}{12} S_B$.

Warm-up example: spaces of holomorphic differentials

- $H_g(m_1, \ldots, m_n)$ with $m_1 + \cdots + m_n = 2g - 2$ - space of pairs (C, w) where w - holomorphic 1-form with zeros of multiplicity $m_1, \ldots, m_n$. Dimension is $2g + n - 1$; for $n = 2g - 2$ dimension equals $4g - 3$.
- Homological coordinates on $H_g(m_1, \ldots, m_n)$: $\oint_{a_\alpha} w$, $\oint_{b_\alpha} w$; $\int_{l_i} w$, $l_i = [x_1, x_i]$. Here $(a_\alpha, b_\alpha, l_i)$ - basis in relative homologies $H_1(C, \{x_k\})$.
- Dual basis in $H_1(C \setminus \{x_k\})$: $(b_\alpha, -a_\alpha, s_k)$. Here s_k - small positively oriented contour around x_m.

Variational formulas

- For basic holomorphic differentials:

$$\frac{\partial v_\alpha(x)}{\partial(\int_{c_i} w)} = \int_{c_i^*} \frac{v_\alpha(y)B(x,y)}{w(y)}$$

- For canonical bimeromorphic differential:

$$\frac{\partial B(x,y)}{\partial(\int_{c_i} w)} = \frac{1}{2\pi i}\int_{c_i^*} \frac{B(x,y)B(x,z)}{w(z)}$$

- For Bergman projective connection:

$$\frac{\partial S_B(x)}{\partial(\int_{c_i} w)} = \frac{1}{12\pi i}\int_{c_i^*} \frac{B^2(x,z)}{w(z)}$$

Bergman tau-function

- Introduce another projective connection:

$$S_w = \frac{w''}{w} - \frac{3}{2}\left(\frac{w'}{w}\right)^2$$

- Define tau-function $\tau(C, w, \{a_\alpha, b_\alpha\})$:

$$\frac{\partial \log \tau(C, w)}{\partial(\int_{c_i} w)} = -\frac{2}{\pi i}\int_{c_i^*} \frac{S_B - S_w}{w}$$

  Compatibility follows from variational formulas for S_B.
- Meaning of τ: $\tau = Z^{24}$, where Z - chiral partition function of free bosons on C.

Properties of the tau-function

- Under transformation of canonical basis of cycles by symplectic matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ τ transforms as follows:

$$\tau(C, w, \{a'_\alpha, b'_\alpha\}) = \det{}^{24}(C\mathbf{B} + D)\,\tau(C, w, \{a_\alpha, b_\alpha\})$$

  where $\mathbf{B}$ - matrix of b-periods of C.
- Under rescaling of w:

$$\tau(C, \epsilon w) = \epsilon^{2\left(2g-2+n-\sum_{k=1}^{n}\frac{1}{m_k+1}\right)}\,\tau(C, w)$$

- Therefore, on open part of projectivized moduli space τ is a section of line bundle

$$\lambda^{24} \otimes L^{-2\left(2g-2+n-\sum_{k=1}^{n}\frac{1}{m_k+1}\right)}$$

  where λ - Hodge line bundle, L-tautological line bundle

Explicit formula for tau-function

$$\tau(C, w, \{a_\alpha, b_\alpha\}) = \frac{\left(\left(\sum_{i=1}^{g} v_i(\zeta) \frac{\partial}{\partial s_i} \right)^g \theta(s; \mathbf{B}) \Big|_{s=K^\zeta} \right)^{16}}{W(\zeta)^{16}}$$

$$\times \frac{\prod_{k<l} E(x_k, x_l)^{4m_k m_l}}{\prod_k E(\zeta, x_k)^{8(g-1)m_k}} w^{8(g-1)}(\zeta)$$

where K^ζ - vector of Riemann constants, $E(x, y)$ - prime-form, W - Wronskian determinant of normalized holomorphic differentials.

For genus 1 the tau-function coincides with Dedekind eta-function:

$$\tau(A, B) = \eta^{48}(B/A)$$

where A and B are periods of w.

Tau-function on compactification of $H_g(1,\dots,1)$

Boundary components of $H_g(1,\dots,1)$:

- D_{deg} - two zeros of w merge to form zero of second order
- Component D_0 of Deligne-Mumford boundary - pinching of C along homologically non-trivial cycle
- Components D_j of Deligne-Mumford boundary, $j=1,\dots,[g/2]$ - pinching of C along homologically trivial cycle to get two Riemann surfaces of genera j and $g-j$.

Asymptotics of τ near all boundary components can be found explicitly, which gives the formula for Hodge class in terms of classes of bondary divisors and tautological class ψ in rational Picard group of projectivization of $H_g(1,\dots,1)$:

$$\lambda=\frac{g-1}{4}\psi+\frac{1}{24}\delta_{deg}+\frac{1}{12}\delta_0+\frac{1}{8}\sum_{j=1}^{[g/2]}\delta_j$$

Spaces of holomorphic quadratic differentials

- $H_{g,2}$ - space of pairs (C, q), where q - holomorphic quadratic differential with simple zeros.
- $\dim H_{g,2} = 6g - 6$
- For fixed C, dimension of linear vector space V_2 of holomorphic quadratic differentials equals $3g - 3$. Denote corresponding vector bundle over moduli space of Riemann surfaces by Λ_2.
  Determinant line bundle: $\lambda_2 = \det \Lambda_2$.
- Relations between classes of λ_2 and λ_1 (Mumford 1977):

$$\lambda_2 - 13\lambda_1 = -\Delta$$

where $\Delta = \sum_{j=0}^{[g/2]} D_j$.

Canonical covering

- $\hat{C}$ - double covering of C; on $\hat{C}$ we have that $w = q^{1/2}$ is a well-defined holomorphic 1-form. Branch points of $\hat{C}$ - zeros x_i ($i = 1, \ldots, 4g-4$) of W. Genus $\hat{g}$ of $\hat{C}$ equals $4g-3$
- w - holomorphic 1-form on $\hat{C}$ with zeros of order 2 at x_i: $w \in H_{\hat{g}}(2, \ldots, 2)$.
- Denote the involution on $\hat{C}$ by $*$. Under the action of $*$ we have the splitting of homologies

$$H_1 = H_+ \oplus H_-$$

where $\mathrm{dim} H_+ = 2g$ and $\mathrm{dim} H_- = 6g-6$.

Isomorphizm between H^- and V_2

- Holomorphic part of cohomologies:

$$H^{(1,0)} = H^+ \oplus H^-$$

where $\dim H^+ = g$ and $\dim H^- = 3g - 3$.

- Isomorphizm between V_2 and H_-:

$$q \quad \in \quad V_2 \qquad \text{then} \qquad v = \frac{q}{w} \quad \in \quad H^-$$

- The link between corresponding determinant line bundles: λ_2 and λ_-:

$$\lambda_- = \lambda_2 - \frac{3}{2}(g-1)\psi$$

Tau-functions on spaces of quadratic differentials

- $\hat{B}(x,y)$ - canonical bimeromorphic differential on $\hat{C}$;

$$B_{\pm}(x,y) = \hat{B}(x,y) \pm \hat{B}(x,y^*) \ .$$

- Corresponding projective connections $S_B^{\pm}$
- Homological coordinates: $\int_s w$ for $s \in H_-$ (exactly $6g-6$ independent).
- Tau-functions $\tau_{\pm}$:

$$\frac{\partial \log \tau_{\pm}(C,q)}{\partial(\int_s w)} = -\frac{2}{\pi i}\int_{s^*} \frac{S_B^{\pm} - S_w}{w}$$

where s^* - cycle dual to s.

Properties of $\tau_\pm$

- Relation with $\hat{\tau}$:
$$\tau_+\tau_- = \hat{\tau}^2(\hat{C}, w)$$
where $\hat{\tau}$ - tau-function on $H_{\hat{g}}(2,\ldots,2)$.
- On open part of $M_{2,g}$ the tau-function $\tau_\pm$ is a section of line bundle
$$\lambda_\pm \otimes L^{\kappa_\pm}$$
where $\kappa_+ = 5/36(g-1)$; $\quad \kappa_- = 11/36(g-1)$ and L is the tautological line bundle.

Line bundle λ_- on compactification of $M_{g,2}$

- Boundary of $M_{g,2}$: 1. Deligne-Mumford boundary Δ of M_g; 2. D_{deg} on divisor D_{deg} two zeros of q merge.
- Asymptotics of $\tau_\pm$ near the boundary implies

$$\lambda_+ = \frac{5(g-1)}{36}\psi + \frac{1}{72}\delta_{deg} + \frac{1}{12}\delta$$

$$\lambda_- = \frac{11(g-1)}{36}\psi + \frac{13}{72}\delta_{deg} + \frac{1}{12}\delta$$

Mumford's formula from tau-functions

- Excluding D_{deg} we get

$$\lambda_- - 13\lambda_+ = -\delta - \frac{3(g-1)}{2}\psi$$

- Using the link between λ_- and λ_2, we get Mumford's relation

$$\lambda_2 - 13\lambda_1 = -\delta$$

Moduli spaces of punctured Riemann surfaces and Strebel differentials

- $M_{g,n}$ - moduli space of genus g Riemann surfaces with n punctures
- Hodge classes: $\lambda_k = c_k(\Lambda)$ - Chern classes of Hodge vector bundle, k=1,. . . ,g
- "psi-classes": $\psi_i, \quad i = 1, \ldots, n$ (fiber: tangent space at ith marked point)
- kappa-classes (Morita-Miller classes) κ_i, $i = 1, \ldots, 3g - 3 + n$
- Relation (Mumford):

$$\kappa_1 = 12\lambda_1 + \psi_1 + \cdots + \psi_n$$

- Arbarello-Cornalba (Penner): κ_1 via Witten's cycle W_5:

$$12\kappa_1 = W_5 + \delta$$

- Hodge class via W_5:

$$\lambda_1 + \frac{1}{12}(\psi_1 + \cdots + \psi_n) = \frac{1}{144}W_5 + \frac{13}{144}\delta$$