Algebro-geometric solutions of the Schlesinger system from conformal field theory

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based on arXiv: 1401.6104

The Fuchsian system

The Fuchsian system

$$\frac{\partial}{\partial y} Y(y) = A(y) Y(y) ,$$

where

$$A(y) = \sum_{k=1}^{n} \frac{A_k}{y - z_k} ,$$

is defined on the sphere $\mathbb{CP}^1$ with n punctures. We will assume $A_1, \ldots A_n \in \mathfrak{sl}(2, \mathbb{C})$, $\sum_{k=1}^{n} A_k = 0$. The fundamental matrix solution $Y(y)$ has non-trivial monodromies around the punctures z_k. Fix starting point y_0 and normalize $Y(y_0) = 1$. The fundamental group $\pi_1(C_{0,n}, y_0)$ of $C_{0,n} := \mathbb{CP}^1 \setminus \{z_1, \ldots, z_n\}$ has n generators $\chi_1, \ldots, \chi_n$ subject to one relation $\chi_1 \circ \chi_2 \circ \cdots \circ \chi_n = 1$. The monodromy along χ_k is represented as

$$Y(\chi_k . y) = Y(y) M_k ,$$

where $Y(\chi_k . y)$ denotes the analytic continuation of $Y(y)$ along χ_k. We come to the anti-homomorphism $\rho : \pi_1(C_{0,n}) \to \mathrm{SL}(2, \mathbb{C})$.

The Riemann-Hilbert problem

The Riemann-Hilbert problem is an inverse problem:

Is it possible to reconstruct the connection $A(y)$ if we specified $\rho : \pi_1(C_{0,n}) \to \mathrm{SL}(2, \mathbb{C})$?

For the case of $\mathrm{SL}(2, \mathbb{C})$, the answer is 'yes'.

We come to the Riemann-Hilbert correspondence between flat connections $\partial_y - A(y)$ and representations $\rho : \pi_1(C_{0,n}) \to \mathrm{SL}(2, \mathbb{C})$. It allows us to identify the moduli space $\mathcal{M}_{\mathrm{flat}}(C_{0,n})$ of flat $\mathfrak{sl}(2, \mathbb{C})$-connections on $C_{0,n}$ with the so-called character variety $\mathrm{Hom}(\pi_1(C_{0,n}), \mathrm{SL}(2, \mathbb{C}))/\mathrm{SL}(2, \mathbb{C})$.

The quotient by $\mathrm{SL}(2, \mathbb{C})$ is used to exclude the dependence on the initial point y_0 by $\mathrm{SL}(2, \mathbb{C})$-conjugation.

We will be interested in the cases where the matrices M_k are diagonalizable with fixed eigenvalues $e^{\pm 2\pi i m_k}$. The space of all such representations of $\pi_1(C_{0,n})$ is then $2(n-3)$-dimensional.

Isomonodromic deformation. Schlesinger equation

It is well-known that variations of the positions $z = (z_1, \ldots, z_n)$ will not change the monodromies of the connection

$$A(y|z) = \sum_{k=1}^{n} \frac{A_k}{y - z_k}$$

provided that the matrix residues $A_k = A_k(z)$ satisfy the the Schlesinger equations

$$\partial_{z_k} A_k = -\sum_{l \neq k} \frac{[A_k, A_l]}{z_k - z_l}, \qquad \partial_{y_0} A_k = -\sum_{l \neq k} \frac{[A_l, A_k]}{y_0 - z_l}.$$

$$\partial_{z_l} A_k = \frac{y_0 - z_k}{y_0 - z_l} \frac{[A_k, A_l]}{z_k - z_l}, \quad k \neq l,$$

Isomonodromic tau-function

The Schlesinger equations define Hamiltonian flows, generated by the Hamiltonians

$$H_k := \frac{1}{2} \operatorname*{\mathit{res}}_{y=z_k} \mathit{tr}\, A^2(y) = \sum_{l \neq k} \frac{\mathrm{tr}(A_k A_l)}{z_l - z_k} \,,$$

using the Poisson structure

$$\{\, A(y) \overset{\otimes}{,} A(y') \,\} \;=\; \left[\frac{\mathcal{P}}{y - y'} \,,\, A(y) \otimes 1 + 1 \otimes A(y') \right],$$

where $\mathcal{P}$ denotes the permutation matrix. The tau-function $\tau(z)$ is defined as the generating function for the Hamiltonians H_k,

$$H_k \;=\; \partial_{z_k} \log \tau(z) \,.$$

Virasoro algebra and its Verma modules

The Virasoro algebra is a Lie algebra with generators L_n, $n \in \mathbb{Z}$ and relations

$$[\, L_n \,,\, L_m \,] \,=\, (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}\,.$$

Highest weight representations $\mathcal{V}_\alpha$ are generated from vectors $|\alpha\rangle$ which satisfy

$$L_n \,|\, \alpha\,\rangle \,=\, 0\,, \quad n > 0\,, \quad L_0 \,|\, \alpha\,\rangle \,=\, \Delta_\alpha \,|\, \alpha\,\rangle\,,$$

where $\Delta_\alpha = \alpha(Q-\alpha)$ if c is parameterized as $c = 1 + 6Q^2$.

The representations $\mathcal{V}_\alpha$ can be decomposed into the so-called energy-eigenspaces

$$\mathcal{V}_\alpha \,\simeq\, \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathcal{V}_\alpha^{(n)}\,,$$

defined by the condition $L_0 v = (\Delta_\alpha + n)v$ for all $v \in \mathcal{V}_\alpha^{(n)}$.

Chiral vertex operators

Chiral vertex operators $V^{\alpha}_{\beta_2\beta_1}(z)$ can be defined as operators that map $\mathcal{V}_{\beta_1} \to \mathcal{V}_{\beta_2}$ such that

$$L_n\, V^{\alpha}_{\beta_2\beta_1}(z) - V^{\alpha}_{\beta_2\beta_1}(z)\, L_n \;=\; z^n(z\partial_z + \Delta_\alpha(n+1))\, V^{\alpha}_{\beta_2\beta_1}(z)\,.$$

We have in particular

$$V^{\alpha}_{\beta_2\beta_1}(z)\,|\,\beta_1\,\rangle \;=\; N(\beta_2,\alpha,\beta_1)\, z^{\Delta_{\beta_2}-\Delta_{\beta_1}-\Delta_\alpha}\,\big[\,|\,\beta_2\,\rangle + (\ldots)L_{-1}|\,\beta_2\,\rangle + \cdots\big]\,,$$

with a normalization factor $N(\beta_2,\alpha,\beta_1)$ that will be specified later. It is well-known that the definition of vertex operator fixes $z^{\Delta_{\beta_1}+\Delta_\alpha-\Delta_{\beta_2}}\, V^{\alpha}_{\beta_2\beta_1}(z)$ uniquely in the sense of formal power series in z,

$$V^{\alpha}_{\beta_2\beta_1}(z) \;=\; z^{\Delta_{\beta_2}-\Delta_{\beta_1}-\Delta_\alpha}\sum_{n=-\infty}^{\infty} z^n\, W^{\alpha}_{\beta_2\beta_1}(n)\,, \qquad W^{\alpha}_{\beta_2\beta_1}(n) : \mathcal{V}^{(k)}_{\beta_1} \to \mathcal{V}^{(k+n)}_{\beta_2}\,.$$

Conformal blocks

The composition $V^{\alpha_2}_{\beta_3\beta_2}(z)V^{\alpha_1}_{\beta_2\beta_1}(w)$ of such vertex operators exists for $|w/z| < 1$, and that matrix elements such as

$$\langle\, \alpha_n \,|\, V^{\alpha_{n-1}}_{\alpha_n\beta_{n-3}}(z_{n-1})\, V^{\alpha_{n-2}}_{\beta_{n-3}\beta_{n-4}}(z_{n-2}) \cdots V^{\alpha_2}_{\beta_1\alpha_1}(z_2) \,|\, \alpha_1 \,\rangle\,,$$

are represented by absolutely convergent power series in z_k/z_{k+1}, $k = 2, \ldots, n-2$.
Such matrix elements are called conformal blocks.

Solution of the Riemann-Hilbert problem in terms of conformal blocks

We shall now specialize to $c = 1$. For that case we shall replace the parameters α_k and β_r by variables m_k and p_r as

$$\alpha_k = im_k, \qquad \beta_r = ip_r,$$

giving the conformal dimensions as $\Delta_{m_k} = m_k^2$ and $\Delta_{p_r} = p_r^2$, for $k = 1, \ldots, n$ and $r = 1, \ldots, n-3$, respectively.
Take normalization of chiral vertex operators as

$$N(p_3, p_2, p_1) =$$
$$= \frac{G(1 + p_3 - p_2 - p_1)G(1 + p_1 - p_3 - p_2)G(1 + p_2 - p_1 - p_3)G(1 + p_3 + p_2 + p_1)}{G(1 + 2p_3)G(1 - 2p_2)G(1 - 2p_1)}$$

where $G(p)$ is the Barnes G-function that satisfies $G(p + 1) = \Gamma(p)G(p)$.

Isomonodromic tau-function in terms of conformal blocks

Theorem: The isomonodromic tau-function is given by Fourier transformation of conformal blocks

$$\tau(z) =$$

$$= \sum_{\vec{k}\in\mathbb{Z}^{n-3}} \prod_{r=1}^{n-3} e^{ik_r q_r} \left\langle m_n \,|\, V^{m_{n-1}}_{m_n\,,\,p_{n-3}+k_{n-3}}(z_{n-1}) \ldots V^{m_3}_{p_2+k_2, p_1+k_1}(z_3)\, V^{m_2}_{p_1+k_1, m_1}(z_2) \,|\, m_1 \right\rangle .$$

The summation runs over the vectors $\vec{k} = (k_1, \ldots, k_N)$ in $\mathbb{Z}^N$. Also we fixed $z_1 = 0$, $z_n = \infty$.

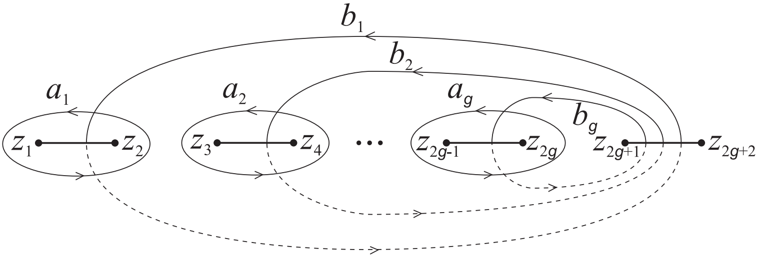
The tau-function depends on $2(n-3)$ parameters $p_1, \ldots, p_{n-3}$, $q_1, \ldots, q_{n-3}$ which are Darboux coordinates on the moduli space of flat $\mathfrak{sl}(2, \mathbb{C})$-connections on $C_{0,n}$.

Conformal blocks of Ashkin-Teller model

Conformal blocks of this form with $m = m_{\rm AT} \equiv \left(\frac{1}{4}, \ldots, \frac{1}{4}\right)$ describe correlation functions of the Ashkin-Teller critical model (Al.Zamolodchikov'87). They can be expressed in terms of certain quantities associated to the hyperelliptic curve Σ of genus g defined by

$$\lambda^2 = \prod_{k=1}^{2g+2} (y - z_k)\,.$$

Let us fix the canonical homology basis of a- and b-cycles on Σ as shown in Figure.



Conformal blocks of Ashkin-Teller model

The g-dimensional space of holomorphic 1-forms on Σ is spanned by

$$d\omega_k = \frac{y^{k-1}dy}{\lambda}, \qquad k = 1, \ldots, g.$$

The $g \times g$ matrices of a- and b-periods

$$A_{jk} = \oint_{a_k} d\omega_j, \qquad B_{jk} = \oint_{b_k} d\omega_j,$$

determine the symmetric period matrix $\Omega = A^{-1}B$ of Σ.

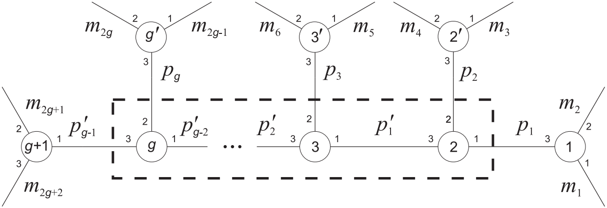
The Riemann theta function with characteristics $[\,p, q\,] \in \mathbb{C}^{2g}$ is defined as the following series:

$$\theta[\,p, q\,]\big(x\,|\,\Omega\big) = \sum_{n \in \mathbb{Z}^g} e^{\pi i (n+p)\cdot\Omega\cdot(n+p) + 2\pi i (n+p)\cdot(x+q)}.$$

Even characteristics $[\,p_S, q_S\,]$ correspond to its non-trivial half-periods and are indexed by partitions $S = \big\{\{z_{\alpha_1}, \ldots, z_{\alpha_{g+1}}\}, \{z_{\beta_1}, \ldots, z_{\beta_{g+1}}\}\big\}$ of the set of ramification points into two subsets of equal size.

Conformal blocks of Ashkin-Teller model

The Ashkin-Teller conformal block has graphical presentation



where the conservation of charge is imposed: $p'_k = p'_{k-1} + p_{k+1}$.

In this notation, the Ashkin-Teller conformal block is given by

$$\mathcal{B}\left(m_{\mathrm{AT}} \,|\, p, p'(p) \,|\, z\right) = \mathcal{G}(p)\mathcal{K}(z)\, \frac{e^{i\pi p\cdot\Omega\cdot p}}{\theta[\,p_S, q_S\,](0\,|\,\Omega)},$$

$$\mathcal{G}(p) = \frac{\cos \pi p'_{g-1}}{\pi^{1-g/2}} \frac{\hat{G}^2(p'_{g-1} + \frac{1}{2})}{\prod_{k=1}^{g} \hat{G}^2(p_k)}, \qquad \hat{G}(p) = \frac{G(1+p)}{G(1-p)},$$

$$\mathcal{K}(z) = \left(\frac{\prod_{j<k}^{g+1}(z_{\alpha_j} - z_{\alpha_k}) \prod_{j<k}^{g+1}(z_{\beta_j} - z_{\beta_k})}{\prod_{j,k}^{g+1}(z_{\alpha_j} - z_{\beta_k})} \right)^{\frac{1}{8}}.$$

Algebro-geometric solution of Schlesinger system

The sum of conformal blocks reduces to the theta function series, so that

$$\tau(z) = \text{const} \cdot \mathcal{K}(z)\, \frac{\theta[\,p,q\,](0\,|\,\Omega)}{\theta[\,p_S,q_S\,](0\,|\,\Omega)}\,,$$

$$\mathcal{K}(z) = \left(\frac{\prod_{j<k}^{g+1}(z_{\alpha_j}-z_{\alpha_k})\prod_{j<k}^{g+1}(z_{\beta_j}-z_{\beta_k})}{\prod_{j,k}^{g+1}(z_{\alpha_j}-z_{\beta_k})}\right)^{\frac{1}{8}}\,.$$

We thus reproduce the $2g$-parameter family of tau functions found by Kitaev and Korotkin in 1998.

The elliptic case $g=1$ corresponds to Picard solutions of Painlevé VI.

Discussion

- We have obtained explicit formulas for $\mathfrak{sl}(2, \mathbb{C})$ isomonodromic tau-function as a Fourier transformation of conformal blocks with respect to intermediate dimensions of conformal blocks.
- Insertion of additional chiral vertex operators corresponding to reducible Virasoro Verma modules allows to construct explicit solution for the linear Fuchsian system.
- It is natural to ask about the generalization of the presented construction to the higher rank Fuchsian systems. They should be related to the conformal blocks of W-algebras instead of Virasoro algebra.

THANK YOU FOR YOUR ATTENTION!