Two explicit examples of non-Abelian monopole

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Bogomolny equation

Bogomolny equations: $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$D_i\Phi = \pm \sum_{j,k=1}^{3} \epsilon_{ijk} F_{jk}.$$

Here F_{ij} Yang-Mills field strength

$$F_{ij} = \partial_i a_j - \partial_j a_i + [a_i, a_j],$$

a_i gauge field, D_i covariant derivative acting on the Higgs field Φ

$$D_i\Phi = \partial_i\Phi + [a_i, \Phi]$$

The boundary conditions supposed to be

$$\left.\sqrt{-\frac{1}{2}\mathrm{Tr}\ \Phi(\boldsymbol{x})^2}\right|_{r\to\infty} \sim 1 - \frac{n}{2r} + O(r^{-2}), \qquad r = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

We consider two cases $n = 2$ and $n = 1$

ADHMN theorem

The charge n monopole solution is given

$$\begin{aligned}\Phi(\boldsymbol{x})_{ab} &= \imath \int_{-1}^{1} z \boldsymbol{v}_a^\dagger(z,\boldsymbol{x}) \boldsymbol{v}_b(z,\boldsymbol{x},)\mathrm{d}z, \\ a_i(\boldsymbol{x})_{ab} &= \imath \int_{-1}^{1} \boldsymbol{v}_a^\dagger(z,\boldsymbol{x}) \frac{\partial}{\partial x_i} \boldsymbol{v}_b(z,\boldsymbol{x})\mathrm{d}z, \quad i = 1,2,3,\end{aligned} \qquad a,b = 1,2.$$

$\boldsymbol{v}_{1,2}(z,\boldsymbol{x})$ – two orthonormalizable

$$\int_{-1}^{1} \boldsymbol{v}_\mu^\dagger(z,\boldsymbol{x}) \boldsymbol{v}_\nu(z,\boldsymbol{x})\mathrm{d}z = \delta_{\mu\nu}, \quad \mu,\nu = 1,2$$

solutions to the **Weyl equation**

$$\left(-\imath 1_{2n} \frac{\mathrm{d}}{\mathrm{d}z} + \sum_{j=1}^{3} (T_j(z) + \imath x_j 1_n) \otimes \sigma_j \right) \boldsymbol{v}(z,\boldsymbol{x}) = 0.$$

ADHMN theorem

$$\left(-\imath 1_{2n}\frac{\mathrm{d}}{\mathrm{d}z}+\sum_{j=1}^{3}(T_j(z)+\imath x_j 1_n)\otimes\sigma_j\right)\boldsymbol{v}(z,\boldsymbol{x})=0.$$

$n\times n$ matrices $T_j(z)$, $z\in(-1,1)$ satisfy to the **Nahm equations**

$$\frac{\mathrm{d}\,T_i(s)}{\mathrm{d}s}=\frac{1}{2}\sum_{j,k=1}^{3}\epsilon_{ijk}[T_j(z),\,T_k(z)],$$

Matrices $T_i(z)$ **Nahm data**

- satisfy to the conditions

$$T_i(z)=-T_i^{\dagger}(z),\quad T_i(z)=T_i^{\dagger}(1-z)$$

- Solutions has simple poles at $z=\pm 1$ with residues

$$\mathrm{Res}_{\pm 1}\,T_i(z),$$

being n-dimensional representation of the gauge group $su(2)$

Hitchin construction (1982,1983)

Nahm equations admit the **Lax form**:

$$\frac{\mathrm{d}L(z,\zeta)}{\mathrm{d}z} = [L(z,\zeta), M(z,\zeta)]$$

L, M are expressible in T_1, T_2, T_3.
Condition

$$\det(L(z,\zeta) - \eta 1_n) = 0$$

yields the curve $\widehat{\mathcal{C}} = (\zeta, \eta)$ of genus

$$g_{\widehat{\mathcal{C}}} = (n-1)^2$$

is the spectral curve of the n-charge of monopole

$$\eta^n + \alpha_1(\zeta)\eta^{n-1} + \ldots + \alpha_n(\zeta) = 0.$$

$\alpha_k(\zeta)$- polynomials in ζ of degree $2k$.

We are considering $n = 3$ and $n = 2$ and curve of genera 4 and 1

Riemann θ-function of an algebraic curve

$$\theta(\boldsymbol{z};\tau) = \sum_{\boldsymbol{n}\in\mathbb{Z}^g} \exp\left\{\imath\pi\boldsymbol{n}^T\tau\boldsymbol{n} + 2\imath\pi\boldsymbol{n}^T\boldsymbol{z}\right\}$$

Homology basis - $2g$ cycles

$$(\mathfrak{a}_1,\ldots,\mathfrak{a}_g;\mathfrak{b}_1,\ldots,\mathfrak{b}_g)$$

Homorphic differentials

$$u_1,\ldots,u_g,\quad u_k = \frac{\eta^i\zeta^j\mathrm{d}\zeta}{\partial_\eta f(\zeta,\eta)}$$

Periods matrices

$$A = \left(\oint_{\mathfrak{a}_k} u_i\right)_{i,k=1,\ldots,g},\quad B = \left(\oint_{\mathfrak{b}_k} u_i\right)_{i,k=1,\ldots,g},\quad \tau = A^{-1}B$$

Hitchin constraints

The curve $\widehat{\mathcal{C}}$ is subjected to the constraints

H1. $\widehat{\mathcal{C}}$ admits the involution

$$(\zeta, \eta) \to \left(-1/\overline{\zeta}, -\overline{\eta}/\overline{\zeta}^2\right)$$

H2. $\mathfrak{b}$-periods of the second kind normalized differentials are half-integer

$$\gamma_\infty(P)_{P\to\infty_i} = \left(\frac{\rho_i}{\xi^2} + O(1)\right)\mathrm{d}\xi, \quad \oint_{\mathfrak{a}_k} \gamma_\infty = 0,$$

$$\boldsymbol{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \dots, \oint_{\mathfrak{b}_n}\gamma_\infty\right)^T = \frac{1}{2}\boldsymbol{n} + \frac{1}{2}\tau\boldsymbol{m},$$

$\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^g$- **Ercolani-Sinha vectors** [Ercolani-Sinha (1989)].

H3. The linear winding $\boldsymbol{U}(z+1) + \boldsymbol{K}$, $\boldsymbol{K}$- vector of Riemann constants, does not intersect theta-divisor, Θ, i.e.:

$$\theta(\boldsymbol{U}(z+1) + \boldsymbol{K}; \tau) \neq 0, \quad z \in (-1, 1).$$

Monopole curve

The curve

$$\eta^n + \alpha_2(\zeta)\eta^{n-2} + \ldots + \alpha_n(\zeta) = 0.$$

satisfying Hitchin constraints **H1**, **H2**. **H3** called monopole curve

Number of parameters or dimension of the moduli space $\mathcal{M}_m$

$$\mathrm{Dim}\,\mathcal{M}_n = \sum_{r=2}^{n}(2r+1) - n = 4(n-1)$$

Three parts of the problem

Part I To find monopole curve

Part II To integrate Weyl equation

Part III To calculate monopole fields - Higgs field and gauges

Part III done because of the Panagopoulos formulae (1982)

A charge 3 monopole curve

The most general charge $n = 3$ monopole curve, that respects **H1.** is the genus $g = 4$ curve,

$$\eta^3 + \eta(\alpha_0\zeta^4 + \alpha_1\zeta^3 + \alpha\zeta^2 - \overline{\alpha}_1\zeta + \overline{\alpha}_0)$$
$$+\beta\zeta^6 + \beta_1\zeta^5 + \beta_2\zeta^4 + \gamma\zeta^3 - \overline{\beta}_2\zeta^2 + \overline{\beta}_1\zeta - \beta = 0$$

α, β, γ - real.
That was found [Hitchin-Manton-Murray (1995)] that the curve

$$\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0$$

$$b = \pm 5\sqrt{2}, \qquad \chi = -\frac{1}{6}\frac{\Gamma(1/6)\Gamma(1/3)}{2^{1/6}\pi^{1/2}}$$

respect all constraints **H1**, **H2**,**H3**.

Therefore only one point in 8-dimensional moduli space was found!

Tetrahedron

F.Klein, Lectures on the icosahedron

$$\alpha = 0, \quad \gamma = 5\sqrt{2}$$
$$\mathcal{P}(\zeta) = \zeta^6 + 5\sqrt{2}\zeta^3 - 1$$
$$\eta^3 + \mathcal{P}(\zeta) = 0$$

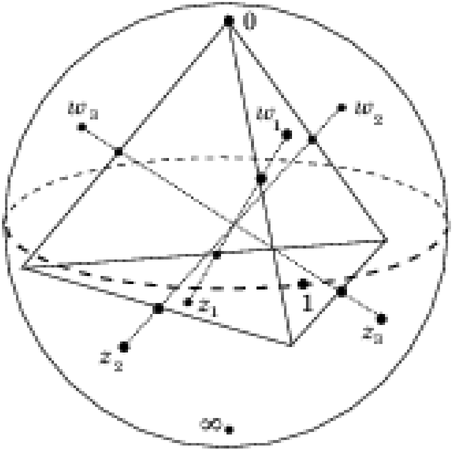


Fig. is taken from G.Hougton, N.Manton and N.Romão, 1999

Symmetric charge 3 monopole curve

The most general charge 3 monopole curve, that respects C_3 symmetry,

$$(\zeta, \eta) \longrightarrow (\rho\zeta, \rho\eta), \quad \rho = \mathrm{e}^{2\imath\pi/3}.$$

$$\eta^3 + \alpha\eta\zeta^2 + \beta\zeta^6 + \gamma\zeta^3 - \beta = 0,$$

where α, β, γ - real. We want to extend above result to this class of curves. To begin we proved:

Theorem [Braden & E, 2009] The class of the monopole curves

$$\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0$$

consists only two representatives,

$$b = \pm 5\sqrt{2}, \qquad \chi = -\frac{1}{6}\frac{\Gamma(1/6)\Gamma(1/3)}{2^{1/6}\pi^{1/2}}$$

Wellstein (1899), Matsumoto (2000)

The curve

$$w^3 = (z-\lambda_1)\dots(z-\lambda_6)$$

Holomorphic differentials

$$\frac{\mathrm{d}z}{w}, \quad \frac{\mathrm{d}z}{w^2}, \quad \frac{z\mathrm{d}z}{w^2}, \quad \frac{z^2\mathrm{d}z}{w^2}.$$

Homology: $\{\mathfrak{a}_1,\dots,\mathfrak{a}_4;\mathfrak{b}_1,\dots,\mathfrak{b}_4\}$. Denote

$$\boldsymbol{X} = \left(\oint_{\mathfrak{a}_1}\frac{\mathrm{d}z}{w},\dots,\oint_{\mathfrak{a}_4}\frac{\mathrm{d}z}{w}\right).$$

Then the period matrix is of the form

$$\widehat{\tau} = \rho^2\left(H + (\rho^2-1)\frac{\boldsymbol{X}\boldsymbol{X}^T}{\boldsymbol{X}^T H\boldsymbol{X}}\right),$$

where $\rho = \exp(2\imath\pi/3)$, $H = \mathrm{diag}(1,1,1,-1)$.

Implementation of Wellstein's result to $\eta^3+\chi(\zeta^6+b\zeta^3-1)=0$.

For a pair of relatively prime integers (m,n) a solution to **H1** and **H2** can be obtained as follows: First solve for t

$$\frac{2n-m}{m+n}=\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1,t\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1,1-t\right)} \quad \Rightarrow \quad b=\frac{1-2t}{\sqrt{t(1-t)}}$$

Ercolani-Sinha vectors and Riemann period matrix are

$$\boldsymbol{n}=\begin{pmatrix} n \\ m-n \\ -m \\ 2n-m \end{pmatrix}, \quad \boldsymbol{m}=\begin{pmatrix} -m \\ n \\ m-n \\ 3n \end{pmatrix}$$

$$\widehat{\tau}=\rho^2 H-(\rho-\rho^2)\frac{(\boldsymbol{n}+\rho^2 H\boldsymbol{m})(\boldsymbol{n}+\rho^2 H\boldsymbol{m})^T}{(\boldsymbol{n}+\rho^2 H\boldsymbol{m})^T H(\boldsymbol{n}+\rho^2 H\boldsymbol{m})}.$$

$$\rho=\mathrm{e}^{2\imath\pi/3}, \quad H=\mathrm{Diag}(1,1,1,-1)$$

Strange equation

Compare our parametrization with Hitchin-Manton-Murray (1995) tetrahedral solution we conclude that at $n = 1$ and $m = 0$ should be:

$$\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-t\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;t\right)} = 2,$$

$$t = \frac{1}{2} - \frac{5\sqrt{3}}{18}, \quad b = 5\sqrt{2}$$

In general: Do other **algebraic** numbers t exist such that

$$\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-t\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;t\right)} = \frac{p}{q} \in \mathbb{Q}, \quad t - \text{algebraic}$$

Ramanujan (1914)

Second Notebook: Let r (signature) and $n \in \mathbb{N}$

$$\frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)} = n\frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-y\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; y\right)}.$$

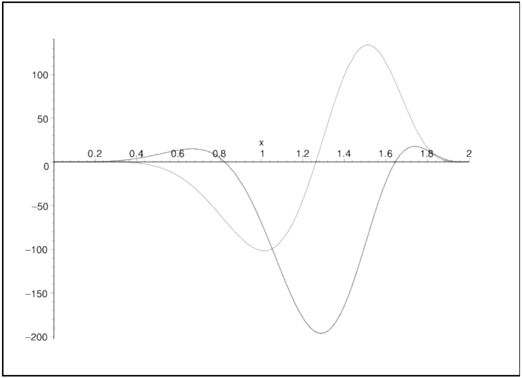
Then $\mathcal{P}(x, y) = 0$ is algebraic equation, find it!
Ramanujan theory for signature 3, $r = 3$, $n = 2$

$$(xy)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}}(1-y)^{\frac{1}{3}} = 1$$

Set $y = \frac{1}{2}$ to obtain $b = 5\sqrt{2}$.
Other signatures: [Berndt & Bhargava & Garvan, 1995]

Tetrahedral monopole exists

Value $b = 5\sqrt{2}$ corresponds to $n = 1, m = 0$ - Check **H3**

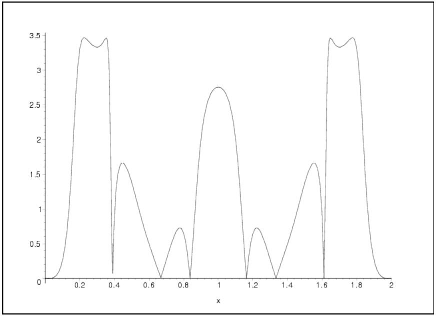


Plot of the real and imaginary parts of the function $\theta(\boldsymbol{U}s + \boldsymbol{K})$, $s \in [0, 2]$

The case $b = -5\sqrt{2}$ is given by $n = m = 1$

Conjecture: No monopoles at other (m,n)

For example, at $n = 4,\ m = -1$



Plot of $|\theta(\boldsymbol{U}s + \boldsymbol{K})|$ and $s \in [0, 2]$. There are 6 additional zeros.
An observation: There are $2(|n| - 1)$ unwanted zeros.

Unramified cover

[Schottky & Jung 1909, Fay 1973]
Our genus 4 curve $\widehat{\mathcal{C}}$ admits automorphism: $\sigma : (\zeta, \eta) \to (\rho\zeta, \rho\eta)$
and covers 3-sheetedly genus 2 curve $\mathcal{C}$.

$$\pi : \widehat{\mathcal{C}} \to \mathcal{C}$$

$$\widehat{\mathcal{C}} : \quad \eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0,$$

$$\mathcal{C} : \ \nu^2 = (\mu^3 + b)^2 + 4$$

$$\nu = \zeta^3 + 1/\zeta^3, \qquad \mu = -\eta/\zeta$$

Riemann-Hurwitz formula,

$$2\widehat{g} - 2 = 2N(g - 1) + B$$

tells that the cover is unramified, $N = 3, \widehat{g} = 4, g = 2 \to B = 0$.

Schottky-Jung proportionality

In the case of unramified cover

$$\pi : \widehat{\mathcal{C}}(\zeta, \eta) \longrightarrow \mathcal{C}(x, y)$$

exists a basis in homology group

$$H(\widehat{\mathcal{C}}, \mathbb{Z}), \quad (\mathfrak{a}_1, \dots, \mathfrak{a}_4; \mathfrak{b}_1, \dots, \mathfrak{b}_4)$$

admitting automorphism σ,

$$\sigma \circ \mathfrak{a}_k = \mathfrak{a}_{k+1}, \quad \sigma \circ \mathfrak{b}_k = \mathfrak{b}_{k+1}, \quad k = 1, 2, 3$$
$$\sigma \circ \mathfrak{b}_0 \sim \mathfrak{b}_0.$$

Factorization of the θ-function

At the above conditions the associated θ-function admits remarkable factorization [Fay-Accola theorem, Fay-73, Eq.67]

$$\frac{\theta(3z_1, z_2, z_2, z_2; \widehat{\tau})}{\theta(z_1, z_2; \tau)\theta(z_1 + 1/3, z_2; \tau)\theta(z_1 - 1/3, z_2; \tau)} = c$$

Here c independent of z_1, z_2, period matrices are

$$\widehat{\tau} = \begin{pmatrix} a & b & b & b \\ b & c & d & d \\ b & d & c & d \\ b & d & d & c \end{pmatrix} \qquad \tau = \begin{pmatrix} \frac{1}{3}a & b \\ b & c + 2d \end{pmatrix}.$$

and $(z_1, z_2) \in \mathrm{Jac}(C)$ is lifted to $\mathrm{Jac}(\widehat{C})$ as

$$(z_1, z_2) \to (3z_1, z_2, z_2, z_2)$$

Remark:

$\widehat{\tau}$ can't be diagonalized or block-diagonalized if the curve is non-singular.

$$\tau \neq \begin{pmatrix} \tau_{11} & 0 & 0 & 0 \\ 0 & \tau_{22} & 0 & 0 \\ 0 & 0 & \tau_{33} & 0 \\ 0 & 0 & 0 & \tau_{44} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \tau' & 0_2 \\ 0_2 & \tau'' \end{pmatrix}$$

The only possibility is quasi-block-diagonal form, in the case exists transformation $\sigma \in \mathrm{Sp}(8, \mathbb{Z})$

$$\sigma \circ \tau = \begin{pmatrix} \tau' & Q \\ Q^T & \tau' \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{pmatrix}$$

Weierstrass Poincaré reduction theory

Humbert variety

Krazer, Lehrbuch der Thetafunktionen, (1903), Belokolos et al., Springer (1994).
Humbert variety H_{h^2}: period matrix τ of genus two curve $\mathcal{C}$ satisfies

$$q_1 + q_2\tau_{11} + q_3\tau_{12} + q_4\tau_{22} + q_5(\tau_{12}^2 - \tau_{11}\tau_{22}) = 0;$$

$$q_i \in \mathbb{Z}, \quad q_3^2 - 4(q_1q_5 + q_2q_4) = h^2, \quad h \in \mathbb{N}.$$

Then exists a symplectic transformation $\mathfrak{S}$

$$\mathfrak{S} : \tau \to \begin{pmatrix} T_1 & \frac{1}{h} \\ \frac{1}{h} & T_2 \end{pmatrix}, \quad h \in \mathbb{N}.$$

Here h - degree of the cover $\mathcal{C}$ over elliptic curve $\mathcal{E}$

$$\pi : \mathcal{C} \to \mathcal{E}.$$

Outline of theta-transformations

$$\widehat{\tau} = \rho^2 H - (\rho - \rho^2)\frac{(\boldsymbol{n} + \rho^2 H\boldsymbol{m})(\boldsymbol{n} + \rho^2 H\boldsymbol{m})^T}{(\boldsymbol{n} + \rho^2 H\boldsymbol{m})^T H(\boldsymbol{n} + \rho^2 H\boldsymbol{m})}.$$

Wellstein

$$\Downarrow$$

$$\begin{pmatrix} a & b & b & b \\ b & c & d & d \\ b & d & c & d \\ b & d & d & c \end{pmatrix}$$

$$\Downarrow$$

Fay-Accola

$$\begin{pmatrix} \frac{1}{3}a & b \\ b & c + 2d \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} T & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{12T} \end{pmatrix}$$

Bolza D_6

H3 condition is reduced to

Proposition [Braden & E, 2009]

$$\theta(\boldsymbol{U}s + \boldsymbol{K}; \tau) = 0 \quad \text{at} \quad s \in (0, 2)$$

iff one from the following **3** conditions satisfies

$$\frac{\vartheta_3}{\vartheta_2}\left(y\sqrt{-3} + \varepsilon\frac{T}{3}|T\right) + (-1)^{\varepsilon}\frac{\vartheta_2}{\vartheta_3}\left(y + \varepsilon\frac{1}{3}|\frac{T}{3}\right) = 0$$

$$\varepsilon = 0, \pm 1, \quad y = \frac{1}{3}s(n+m), \quad T = \frac{2\sqrt{-3}(n+m)}{2n-m}$$

The solution $y = y(T)$ provides the answer.
We reduced problem in $(n, m) \in \mathbb{Z}^2$ to one variable T

A new θ-constant relation ?

$$\frac{\vartheta_3}{\vartheta_2}\left(\frac{\tau}{3}|\tau\right) = \frac{\vartheta_2}{\vartheta_3}\left(\frac{1}{3}|\frac{\tau}{3}\right)$$

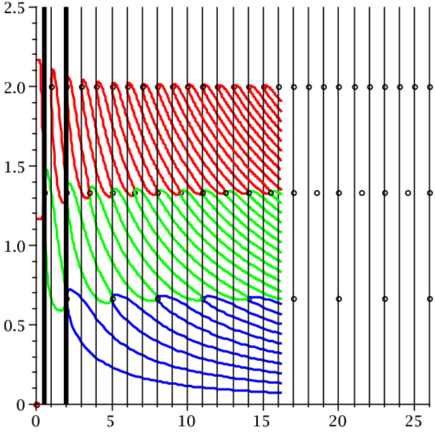
$$\vartheta_4^3(0|\tau)\imath\sqrt{3}\frac{\vartheta_1\left(\frac{\tau}{3}|\tau\right)\vartheta_4\left(\frac{\tau}{3}|\tau\right)}{\vartheta_2\left(\frac{\tau}{3}|\tau\right)^2} + \vartheta_4^2\left(0|\frac{\tau}{3}\right)\frac{\vartheta_1\left(\frac{1}{3}|\frac{\tau}{3}\right)\vartheta_4\left(\frac{1}{3}|\frac{\tau}{3}\right)}{\vartheta_3\left(\frac{1}{3}|\frac{\tau}{3}\right)^2} = 0$$

We are able to prove that using Ramanujan third order transformation of Jacobian moduli

$$k(\tau) \equiv \frac{\vartheta_2(0|\tau)^2}{\vartheta_3(0|\tau)^2} = \frac{(p+1)^3(3-p)}{16p},$$

$$k(\tau/3) \equiv \frac{\vartheta_2(0|\tau/3)^2}{\vartheta_3(0|\tau/3)^2} = \frac{(p+1)(3-p)^3}{16p^3}$$

No charge 3 monopoles beside tetrahedral



Three branches of the function y plotted against $(n+m)/(2n-m)$

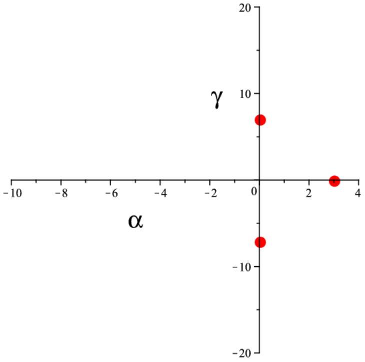
Only two cases $(n+m)/(2n-m) = 2$ and $(n+m)/(2n-m) = 1/2$ satisfy **H3**

Charge 3 monopole curve with cyclic symmetry

The genus four curve $\widehat{\mathcal{C}} = (\zeta, \eta)$ satisfying to **H1**

$$\eta^3 + \alpha\eta\zeta^2 + \zeta^6 + \gamma\zeta^3 - 1 = 0, \qquad \alpha, \gamma \in \mathbb{R}$$

But only 3 points were explicitly known:

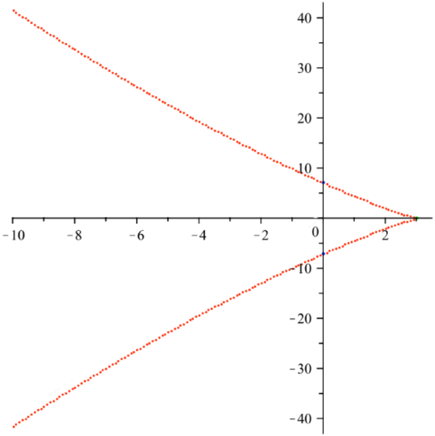


Do other points exist in the (α, γ)-plane?

New monopole curve

The above result can be extended to the curve of genus 4

$$\eta^3 + \alpha\eta\zeta^2 + \beta\zeta^6 + \gamma\zeta^3 - \beta = 0$$



Axis α - horizontal and γ - vertical.

AGM - Arithmetic-Geometric mean

Above genus four curve covers 3-sheetedly the genus two curve

$$y^2 = (x^3 + \alpha x + \gamma)^2 + 4\beta^2$$

and Schottky-Jung factorization is still applicable.
H2 is formulated as a condition on complete holomorphic integrals over this genus two curve:

$$\oint_{\mathfrak{c}} \frac{\mathrm{d}x}{y} = 0, \quad \oint_{\mathfrak{c}} \frac{x\mathrm{d}x}{y} = 6\beta^{1/3}$$

taken along certain cycle $\mathfrak{c}$.

Gauss: AGM (arithmetic-geometric mean)

$$a_0 = a \qquad\qquad b_0 = b$$

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad\qquad b_{n+1} = \sqrt{a_n b_n}$$

The following limit exists and called AGM

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = M(a,b)$$

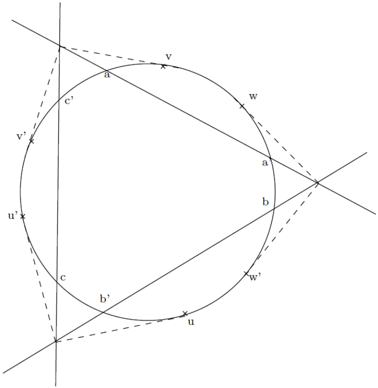
Then (Gauss 1799)

$$\int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} = \frac{\pi}{2M(a,b)}$$

Richelot (1836) and Humbert (1901) generalized AGM to genus two curves

Richelot: AGM for genus two curve

$$
\begin{aligned}
y^2 &= P(x)Q(x)R(x) \\
P(x) &= (x-a)(x-a') \\
Q(x) &= (x-b)(x-b') \\
R(x) &= (x-c)(x-c')
\end{aligned}
$$



$$\int_a^{a'} \frac{S(x)\mathrm{d}x}{\sqrt{-P(x)Q(x)R(x)}} = \pi T \frac{S(\alpha)}{(\alpha-\beta)(\alpha-\gamma)}$$

$$\alpha = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a'_n, \ldots, \gamma = \lim_{n\to\infty} c_n = \lim_{n\to\infty} c'_n$$

Fig. is taken from J.Bost, J.Mestre, 1988

Absolute invariants of the genus two curve

Denote $(i, j) = e_i - e_j$, e_j- branch points of the sextic,

$$y^2 = u_0 x^6 + u_1 x^5 + \ldots + u_6$$

The relative invariants

$$J_2 = u_0^2 \sum_{\text{fifteen}} (12)^2(34)^2(56)^2$$

$$J_4 = u_0^4 \sum_{\text{ten}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2$$

$$J_6 = u_0^6 \sum_{\text{sixty}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2$$

$$J_{10} = u_0^{10} \prod_{j<k} (j, k)^2,$$

3 absolute invariants

$$i_1 = 144 \frac{J_4}{J_2^2}, \quad i_2 = -1728 \frac{(J_4 J_2 - 3J_6)}{J_2^3}, \quad i_3 = 486 \frac{J_{10}}{J_2^5}$$

Humbert variety H_4 again

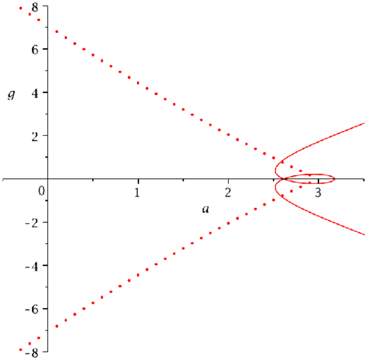
Theorem [van der Geer 1982] Exists such the modular form of the weight 30 denoted χ_{30} that

$$\tau \in H_4, \qquad \text{iff} \qquad \chi_{30} = 0$$

Proposition [Shaska, 2000] The aforementioned form χ_{30} is given explicitly as

$$\chi_{30} = -J_2^7 J_4^4 + 8748 J_{10} J_5^2 + 507384000 J_{10}^2 J_4^2 J_2 - 80 J_4^7 J_2 \\ +30 \text{ further terms}$$

Elliptic points on the new monopole curve



Intersection with $\chi_{30} = 0$

Lesser known Nahm Ansatz

Weyl equation:

$$\left(\imath 1_{2n}\frac{\mathrm{d}}{\mathrm{d}z} - \sum_{j=1}^{3}(T_j(z) + \imath x_j 1_n)\otimes\sigma_j\right)\boldsymbol{v}(z,\boldsymbol{x}) = 0.$$

Construction equation:

$$\left(\imath 1_{2n}\frac{\mathrm{d}}{\mathrm{d}z} + \sum_{j=1}^{3}(T_j(z) + \imath x_j 1_n)\otimes\sigma_j\right)\boldsymbol{w}(z,\boldsymbol{x}) = 0.$$

Fundamental solutions of the Weyl and Construction equations

$$v = \left(\boldsymbol{v}^{(1)}(z,\boldsymbol{x}),\dots,\boldsymbol{v}^{(2n)}(z,\boldsymbol{x})\right)$$
$$w = \left(\boldsymbol{w}^{(1)}(z,\boldsymbol{x}),\dots,\boldsymbol{w}^{(2n)}(z,\boldsymbol{x})\right)$$

are related as

$$v(z,\boldsymbol{x}) = w(z,\boldsymbol{x})^{-1\dagger}$$

Reduction the Weyl equation to *n*-the order ODE

Any column vector of the fundamental solution V is presented in the form – **Nahm Ansatz**

$$\boldsymbol{w} = \left[1_2 + \sum_{k=1}^{3} u_k(\zeta)\sigma_k \right] |s> \otimes \boldsymbol{\Psi}(z,\zeta),$$

where ζ- is certain parameter and the real vector,

$$\boldsymbol{u} = (u_1, u_2, u_3), \quad u_1^2 + u_2^2 + u_3^2 = 1$$

is constructed in terms of the vector $\boldsymbol{y}$,

$$\boldsymbol{y} = \left(\frac{1+\zeta^2}{2\imath}, \frac{1-\zeta^2}{2}, -\zeta \right), \quad \boldsymbol{y}\cdot\boldsymbol{y} = 0$$

$$\boldsymbol{u} = \imath \frac{\boldsymbol{y}\times\boldsymbol{y}}{\boldsymbol{y}\cdot\overline{\boldsymbol{y}}}$$

Substitution of Nahm Ansatz to Construction equation leads

$$\begin{aligned} (L(\zeta) - \eta)\,\boldsymbol{\Psi}(z,\zeta) = 0, \\ \left(\frac{\mathrm{d}}{\mathrm{d}z} + M(z,\zeta)\right)\boldsymbol{\Psi}(z,\zeta) = 0, \end{aligned}$$

with the constraint: – **Atiyah-Ward constraint:**

$$\eta = 2\boldsymbol{y}\cdot\boldsymbol{x},$$

that is $2n$-th order algebraic equation

$$\det(L(\zeta) - 2\boldsymbol{y}\cdot\boldsymbol{x}) = 0.$$

Charge two monopole

Weyl equation

$$\imath 1_4 \frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{w}(z) + \mathcal{F}\boldsymbol{w}(z) = 0$$

$$\mathcal{F} =$$

$$\begin{pmatrix} \frac{1}{2}f_3 + x_3 & x_1 - \imath x_2 & 0 & \frac{1}{2}(f_1 - f_2) \\ x_1 + \imath x_2 & -\frac{1}{2}f_3 - x_3 & \frac{1}{2}(f_1 + f_2) & 0 \\ 0 & \frac{1}{2}(f_1 + f_2) & -\frac{1}{2}f_3 + x_3 & x_1 - \imath x_2 \\ \frac{1}{2}(f_1 - f_2) & 0 & x_1 + \imath x_2 & \frac{1}{2}f_3 - x_3 \end{pmatrix}$$

$$f_1 = \frac{1}{\mathrm{cn}(z/k')}, \quad f_2 = \frac{\mathrm{dn}(z/k')}{k'\mathrm{cn}(z/k')}, \quad f_2 = \frac{\mathrm{sn}(z/k')}{\mathrm{cn}(z/k')}$$

4th order ODE with elliptic function coefficients

Solution in the x_2-direction

[Brown&Prasad&Panagopoulos (1983)]
At $x_1 = x_3 = 0$ the system decompose to two pair of ODE that are reduced to the Lamé equation with one-gap potential.
The Higgs field can be computed analytically:

$$H(x_2) = -k'K + \frac{2k'}{k^2\mathrm{sn}^2 t - S^2}\left(S - \frac{\mathrm{sn}t}{\mathrm{cn}t\,\mathrm{dn}t}\frac{\mathrm{d}S}{\mathrm{d}t}\right)$$

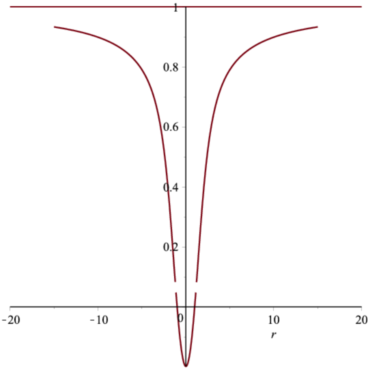
$$S(t) = -\frac{\mathrm{sn}t\,\mathrm{dn}t}{\mathrm{cn}t}\tanh(KZ(t))$$

$$4k'^2 x_2^2 = k^2 \mathrm{sn}t$$

Asymptotic:

$$H(x_2) \sim k'K - \frac{1}{|x_2|} + O\left(\frac{1}{|x_2|^2}\right)\bigg|_{x_2 \to \pm\infty}$$

Higgs field H in $x_2 = r$ direction



Until now Higgs field and gauges are not derived analytically for general $\boldsymbol{x}$ even for this case, $n = 2$

Monopole fields in general case

$$\imath 1_4 \frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{w}(z) - \mathcal{F}\boldsymbol{w}(z) = 0$$

Plug **Nahm Anzats**

$$\boldsymbol{w}(z,\zeta) = \begin{pmatrix} 1 \\ \imath\zeta \end{pmatrix} \otimes \begin{pmatrix} \phi_1(z,\zeta) \\ \phi_2(z,\zeta) \end{pmatrix} \mathrm{e}^{-\imath z[(x_1-\imath x_2)\zeta - \imath x_3]}$$

to get

- Master equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + M(z,\zeta)\right)\phi = 0$$

- Atiyah-Ward constraint

$$4k'^2\left[(\imath x_2 - x_1)\zeta^2 - \imath x_2 - x_1 + 2\imath\zeta x_3\right]^2 + k^2\zeta^4 - 2(k'^2+1)\zeta^2 + k^2 = 0$$

Atiyah-Ward constraint yields four points $\zeta_1, \ldots, \zeta_4$ of the elliptic curve

$$k'^2\eta^2 = k^2\zeta^4 - 2(k'^2+1)\zeta^2 + k^2 = 0$$

General solution of the Master equation is expressible in terms of Bloch functions resolving Lamé equation with one-gap potential,

$$\frac{\mathrm{d}^2\Psi(u,\upsilon)}{\mathrm{d}u^2} - \mathfrak{P}(u)\Psi(u,\upsilon) = \lambda\Psi(u,\upsilon)$$

with one-gap potential $\mathfrak{P}(u)$ and spectral parameter λ of the form

$$\mathfrak{P}(u) = 2\wp(u|\omega,\omega') + 2\frac{(e_1-e_2)(e_1-e_3)}{\wp(u|\omega,\omega')-e_1} \equiv 2\wp\left(u\,\middle|\,\frac{\omega}{2},\omega'\right)$$

$$\lambda = (e_2-e_3)\zeta^2 - 2e_1 \equiv \wp\left(v\,\middle|\,\frac{\omega}{2},\omega'\right)$$

$$z = Kk' - 2k'u, \quad v = \left(\oint_{\mathfrak{a}} \frac{\mathrm{d}\zeta}{\eta}\right)^{-1} \int_{\infty}^{P} \frac{\mathrm{d}\zeta}{\eta}$$

Explicit integration of the Weyl equation in the ADHMN construction

Let $\widehat{\mathcal{C}}$ - monopole curve of genus $g = (n-1)^2$

$$\eta^n + a_1(\zeta)\eta^{n-1} + \ldots + a_n(\zeta) = 0$$

satisfying Hitchin constraint **H1**, **H2**, **H3**. Then monopole fields $\Phi(\boldsymbol{x})$ and $a_j(\boldsymbol{x})$ are expressible in terms of values of **Baker-Akhiezer function**

$$\boldsymbol{\Psi}(\pm 1, P_k(\boldsymbol{x}))$$

at the boundaries of the interval $z = \pm 1$ and algebraic functions of $\boldsymbol{x}$, $P_k(\boldsymbol{x})$ that are solutions of $2n$ algebraic equation, so called **Atiyah-Ward constraint**.