# Coble surfaces, heat equations, and Weierstrass functions in genus 2 and 3

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# Outline




We discuss the properties of generalized $\wp$ and $\sigma$ functions associated with plane curves of genus $g$ (here we consider $g = 2, 3$ only). These functions satisfy interesting nonlinear integrable PDEs, linear parabolic PDEs, and addition formulae. The PDEs can all be be generated from partial derivatives of a single hypersurface in each case. We show also that coefficients of the sigma expansion in genus 3 satisfy a complicated linear recurrence relations.

# Elliptic functions - Genus 1

Elliptic curve in standard Weierstrass form is

$$y^2 = 4x^3 - g_2 x - g_3$$

Associated with this curve is the Weierstrass $\wp(u)$ function, $u = \int dx/y$, satisfying the ODEs

$$\left(\wp'\right)^2 = 4\wp^3 - g_2\wp - g_3,$$
$$\wp'' = 6\wp^2 - \tfrac{1}{2}g_2.$$

The $\sigma(u)$ function is the 2nd logarithmic derivative of $\wp$

$$\wp(u) = -\frac{\partial^2}{\partial u^2} \ln \sigma(u)$$

# $\sigma$ and $\wp$ functions - general $g$

Associated with a curve of genus $g$, there is an *entire* function $\sigma$ of $g$ variables which generalises the Weierstrass $\sigma(u)$ function

$$\sigma(\boldsymbol{u}) = \sigma(u_1, u_2, \ldots, u_g).$$

Using $\sigma$, we can define generalized $\wp$ functions

$$\wp_{ij}(u_1, u_2, \ldots) \equiv -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(u_1, u_2, \ldots).$$

In genus 3, for example, we have three $u_i$ variables and six second derivatives $\wp_{ij}(u_1, u_2, u_3)$: $\wp_{11}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}$.
Higher derivatives are defined in a similar way

$$\wp_{ij\ldots}(u_1, u_2, \ldots) \equiv -\frac{\partial\partial\ldots}{\partial u_i \partial u_j \ldots} \ln \sigma(u_1, u_2, \ldots),$$

# General notation for derivatives

Note that for $g = 1$ in our general $g$ notation, the definition becomes

$$\wp_{11}(u_1) \equiv -\frac{\partial^2}{\partial u_1^2} \ln \sigma(u_1)$$

with the ODEs written as

$$(\wp_{111})^2 = 4\wp_{11}^3 - g_2\wp_{11} - g_3,$$
$$\wp_{1111} = 6\wp_{11}^2 - \tfrac{1}{2}g_2.$$

The initial work in this area was to find the analogue of these differential equations for curves of higher genus. More recently we have investigated their algebraic structure, the addition theorems satisfied by $\sigma$ and $\wp$, and the properties of the $\sigma$ function expansion.

We work *only* with curves with a branch point at infinity, of the form

$$y^n + \dots = x^s + \dots$$

with $n < s$, $n$ and $s$ co-prime, the so-called $(n, s)$ curves.

# Hyperelliptic case, $g = 2$

The general hyperelliptic curve ($n = 2$) takes the form

$$C: \quad y^2 = x^s + \lambda_{s-1}x^{s-1} + \cdots + \lambda_0, \quad s > 4.$$

The simplest case, $s = 5$, has genus 2. It was considered in detail by Baker (1907). $\sigma$ and $\wp$ are now functions of $g = 2$ variables, i.e.

$$\sigma = \sigma(u_1, u_2) = \sigma(\mathbf{u})$$

There are two differentials of the first kind, $dx/y$ and $x\,dx/y$, and we have

$$u_1 = \int^{(x_1,y_1)} \frac{dx}{y} + \int^{(x_2,y_2)} \frac{dx}{y}, \quad u_2 = \int^{(x_1,y_1)} \frac{x\,dx}{y} + \int^{(x_2,y_2)} \frac{x\,dx}{y},$$

for two variable points $(x_i, y_i)$ on $C$. Various techniques (no space to discuss!) then lead to sets of PDEs for the $\wp(u_1, u_2)$ functions.

# Hyperelliptic case, $g = 2$, PDEs

In the genus 2 case, we get five equations expressing the 4th derivative $\wp$ functions $\wp_{ijk\ell}$ in terms of the 2nd derivative $\wp_{ij}$.

$$\begin{aligned}
\wp_{2222} - 6\wp_{22}^2 &= \tfrac{1}{2}\lambda_3 + \lambda_4\wp_{22} + 4\wp_{12}, \qquad \text{(KdV)}\\
\wp_{1222} - 6\wp_{22}\wp_{12} &= \lambda_4\wp_{12} - 2\wp_{11},\\
\wp_{1122} - 2\wp_{22}\wp_{11} - 4\wp_{12}^2 &= \tfrac{1}{2}\lambda_3\wp_{12},\\
\wp_{1112} - 6\wp_{11}\wp_{12} &= \lambda_2\wp_{12} - \tfrac{1}{2}\lambda_1\wp_{22} - \lambda_0,\\
\wp_{1111} - 6\wp_{11}^2 &= \lambda_1\wp_{12} + \lambda_2\wp_{11} - 3\lambda_0\wp_{22} - \tfrac{1}{2}\lambda_0\lambda_4 - \tfrac{1}{8}\lambda_1\lambda_3.
\end{aligned}$$

These are the generalization of $\wp'' - 6\wp^2 = -\frac{1}{2}g_2$ in genus 1 ($\wp_{1111} - 6\wp_{11}^2 = -\frac{1}{2}g_2$).

# Weights in $\wp$ theory

Consider the PDEs, first being

$$\wp_{2222} - 6\wp_{22}^2 = \tfrac{1}{2}\lambda_3 + \lambda_4\wp_{22} + 4\wp_{12},$$

We can *grade* all the equations in the theory by assigning a weight to each term. In the $g = 2$ case, the appropriate weights are

| $u_1$ | $u_2$ | $\lambda_4$ | $\lambda_3$ | $\lambda_2$ | $\lambda_1$ | $\lambda_0$ |
|---|---|---|---|---|---|---|
| 3 | 1 | -2 | -4 | -6 | - 8 | -10 |

The weight of $\sigma$ doesn't count, due to properties of the logarithmic derivatives, so the 4-index PDE above has homogeneous weight -4, and the other 4-index PDEs have weight -6, -8, -10, -12, respectively.

# Hyperelliptic case, $g = 2$, PDEs

In the genus 2 case we have 10 generalizations of the formula $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, ($\wp_{111}^2 = 4\wp_{11}^3 - g_2\wp_{11} - g_3$) in genus 1. The first three are (weights given in square brackets)

$$\begin{aligned}
\wp_{222}^2 &= 4\wp_{22}^3 + 4\wp_{12}\wp_{22} + 4\wp_{11} + \lambda_4\wp_{22}^2 + \lambda_2, \quad [-6] \\
\wp_{122}\wp_{222} &= 4\wp_{22}^2\wp_{12} + \lambda_4\wp_{22}\wp_{12} + 2\wp_{12}^2 - 2\wp_{11}\wp_{22} + \tfrac{1}{2}\lambda_3\wp_{22} + \tfrac{1}{2}\lambda_1, \quad [-8] \\
\wp_{122}^2 &= 4\wp_{12}^2\wp_{22} + \lambda_4\wp_{12}^2 + \lambda_4\wp_{12}^2 - \lambda_0, \quad [-10] \\
\ldots &= \ldots
\end{aligned}$$

There are also 4 PDEs which are *bilinear* in the 3rd and 2nd derivatives $\wp_{ijk}$ and $\wp_{ij}$. These PDEs have no analogue in genus 1

$$\begin{aligned}
\wp_{122} + \wp_{22}\wp_{122} - \wp_{12}\wp_{222} &= 0, \quad [-7] \\
2\wp_{11}\wp_{222} + 2\wp_{111} + (\tfrac{1}{2}\lambda_3 + 2\wp_{12})\wp_{122} - (\lambda_4 + 4\wp_{22})\wp_{112} &= 0, \quad [-9] \\
\cdots &= 0
\end{aligned}$$

Many results for the hyperelliptic case for *arbitrary g* have been developed by Buchstaber, Enolskii, and Leykin.

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# Algebraic viewpoint

We will mostly consider the PDEs as a set of *algebraic* equations for the variables $\wp_{ij}$, $\wp_{ijk}$, etc. It is often convenient to rename these quantities for conciseness, using the (-) weights as subscripts. We define

$$X_2 = \wp_{22}, X_4 = \wp_{12}, X_6 = \wp_{11} = X_6,$$
$$X_9 = \tfrac{1}{2}\wp_{111},\ X_7 = \tfrac{1}{2}\wp_{112},\ X_5 = \tfrac{1}{2}\wp_{122},\ X_3 = \tfrac{1}{2}\wp_{222}.$$

The extra factors of $\frac{1}{2}$ are for later convenience. So the PDE

$$\wp_{222}^2 = 4\wp_{22}^3 + 4\wp_{12}\wp_{22} + 4\wp_{11} + \lambda_4\wp_{22}^2 + \lambda_2,$$

becomes a cubic equation in 4 variables

$$-X_3^2 + X_2^3 + X_2X_4 + X_6 + \lambda_4 X_2^2 + \tfrac{1}{4}\lambda_2 = 0,$$

# Kummer variety, $g = 2$

We can derive the formula for "Kummer's quadratic surface" in the genus two case by noting that

$$(\wp_{222}^2) \cdot (\wp_{122}^2) - (\wp_{122}\wp_{222})^2 = 0$$

We call such relations "Kummer Relations". This one gives a quartic in three variables $X_2 = \wp_{22}, X_4 = \wp_{12}, X_6 = \wp_{11}$, with weight -16.

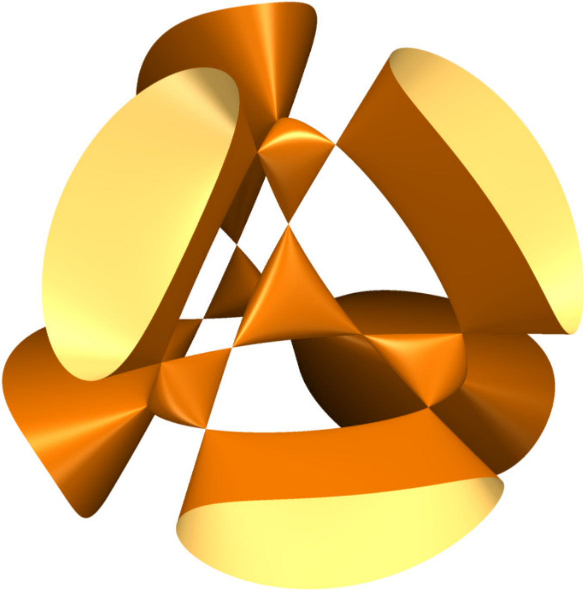
$$\begin{aligned}
&-4\,X_4\lambda_4X_2\lambda_1 - 2\,X_4\lambda_3\lambda_1 - 4\,X_4X_6\lambda_2 + 4\,X_2\lambda_3\lambda_0 + 2\,X_2X_6\lambda_1 + 4\,X_2X_4\lambda_0 \\
&+ 4\,\lambda_2X_2{X_4}^2 + 4\,X_6\lambda_4{X_4}^2 + 4\,\lambda_4{X_2}^2\lambda_0 + 2\,X_6X_2{X_4}^2 + 4\,X_6\lambda_0 + 4\,\lambda_2\lambda_0 \\
&- 4\,X_4{X_6}^2 - 2\,{X_4}^2\lambda_1 + 4\,{X_2}^3\lambda_0 - {X_2}^2{X_6}^2 - 2\,{X_4}^3\lambda_3 - {\lambda_3}^2{X_4}^2 - {\lambda_1}^2 - {X_4}^4 \\
&- 2\,\lambda_3X_2X_4X_6 - 4\,{X_2}^2X_4\lambda_1 + 4\,\lambda_2\lambda_4{X_4}^2 = 0 \qquad (K_{16})
\end{aligned}$$

We can form other cross-products, but all lower weight KRs, down to

$$\wp_{111}^2\wp_{112}^2 - (\wp_{111}\wp_{112})^2 = 0,$$

with weight -32, have $K_{16}$ as a factor.

# Kummer surface, $g = 2$



# Ideals & varieties

The set of 14 PDEs $F_i(X_2, X_3, \dots) = 0$ described above define a *variety*.

A brief recap: an *ideal* generated by a finite set of polynomials $\langle f_1(x_1, x_2, \dots), f_2(x_1, x_2, \dots), \dots, f_n(x_1, x_2, \dots)\rangle$ is the infinite set

$$\begin{aligned} G(x_1, x_2, \dots) =& g_1(x_1, x_2, \dots)f_1(x_1, x_2, \dots) + g_2(x_1, x_2, \dots)f_2(x_1, x_2, \dots) \\ & + \dots + g_n(x_1, x_2, \dots)f_n(x_1, x_2, \dots), \end{aligned}$$

as the $g_i(x_1, x_2, \dots)$ range over *all* possible choices of polynomials.

Clearly, *if* the $f_i$ satisfy

$$f_i(x_1, x_2, \dots) = 0, \quad i = 1, \dots, n, \quad (*)$$

for some values of $x_1, x_2, \dots, x_m$, then *any* $G$ in the ideal will also be zero for these values of the $x_i$. The set of values of $x_i$ satisfying $(*)$ is called a *variety*.

# Ideals and Gröbner Bases

How independent is our set of 14 PDEs as an *algebraic* set of equations? How many of these do we need, to generate an ideal to which all the others belong?

This is like the following problem in linear algebra: given a set of vectors $v_1, v_2, \ldots$, how many are independent? Do they form a complete set? How do we express an arbitrary vector $v$ in terms of the $v_1, v_2, \ldots$? In linear algebra, we can use Gaussian elimination. Remarkably, the corresponding algorithmic technique for polynomials was not discovered until 1965: Gröbner Bases (GB).

Two ideals are equal if they have the same GB. A polynomial lies in the ideal if the remainder on division by all the elements of the GB is zero.

# Coble cubic, $g = 2$?

Suppose $A$ is a variety associated with a curve of genus $g = 2$. We can use 3rd order theta functions to embed $A$ in a projective space $\mathbb{P}(V_3)$ of dimension 8. Coble showed there is a unique *cubic* hypersurface in $\mathbb{P}(V_3)$ that is singular along $A$. The cubic is expressed in terms of theta functions, with transcendental coefficients.

We want to find this Coble cubic. Instead of using theta functions, we will use 3rd order $\wp$ functions. All coefficients will be given *explicitly* in terms of $\lambda_i$. The first step is to find the appropriate variety in the genus 2 case. This is given in Grant (1990).

In genus 2, the 3rd order Abelian functions are $\wp_{11}, \wp_{12}, \wp_{22}, \wp_{111}, \wp_{112}, \wp_{122}, \wp_{222}$, and $\wp_{11}\wp_{22} - \wp_{12}^2$. A key idea in Grant's theory is to use this last function to reduce the Kummer to a cubic.

# Grant's variety in genus 2

We use the following eight variables

$$X_8 = \tfrac{1}{2}(\wp_{11}\wp_{22} - \wp_{12}^2 + \lambda_3\wp_{12} - \lambda_1),$$
$$X_6 = \wp_{11}, \quad X_4 = \wp_{12}, \quad X_2 = \wp_{22},$$
$$X_9 = \tfrac{1}{2}\wp_{111},\ X_7 = \tfrac{1}{2}\wp_{112},\ X_5 = \tfrac{1}{2}\wp_{122},\ X_3 = \tfrac{1}{2}\wp_{222}.$$

As well as the definition of $X_8$, we use two of the bilinear equations given above, plus the three equations for $\wp_{222}^2, \wp_{122}\wp_{222}$, and $\wp_{122}^2$. These six equations define an ideal $J$ and associated variety. We can check that this variety contains $K_{16}$, and *all* other equations in our set of 14.

A key feature is that once we know the weight of the Coble cubic, the number of terms which can contribute is *finite*. We establish this weight by guesswork from the Gröbner basis.

# The Coble cubic in genus 2

We can calculate (Maple) the Gröbner Basis for the ideal $J$ - this has eight quadrics and fourteen cubics. The Coble cubic must vanish on this variety, and also its partial derivatives with respect to the variables and the $\lambda_i$ must vanish. By using the GB, we can find the unique cubic

$$\begin{aligned} C = & {X_2}^3\lambda_0 + \left(\lambda_2 {X_4}^2 + (\lambda_0 - \lambda_4\lambda_1) X_4 - X_8X_6 + X_9X_5 + \lambda_3\lambda_0 - {X_7}^2\right) X_2 \\ & + (\lambda_4\lambda_0 - X_4\lambda_1) {X_2}^2 - \lambda_0 {X_3}^2 + (X_7X_6 - X_9X_4 + \lambda_1 X_5) X_3 - {X_4}^3\lambda_3 \\ & + (\lambda_2\lambda_4 + \lambda_4 X_6 + X_8) {X_4}^2 + \left(X_7X_5 - {X_6}^2 - X_8\lambda_3 - X_6\lambda_2 - \lambda_3\lambda_1\right) X_4 \\ & - (X_6 + \lambda_2) {X_5}^2 + X_7\lambda_3 X_5 + X_9X_7 + {X_8}^2 - {X_7}^2\lambda_4 + X_6\lambda_0 + \lambda_2\lambda_0 + \lambda_1 X_8 \end{aligned}$$

By construction, $C \in J$, also $\partial C/\partial X_i, \partial C/\partial \lambda_i, \in J$, and $J$ can be shown to be identical to $\langle \partial C/\partial X_i, \partial C/\partial \lambda_i \rangle$.

$$\text{Example:} \quad \frac{\partial C}{\partial X_9} = 0 = X_2X_5 - X_3X_4 + X_7 \quad \text{(wt 7 bilinear)}$$

In summary, $C$ encodes *all* our 14 equations and the Kummer surface in a compact form.

# Genus 3, (3, 4) trigonal curve

The simplest trigonal curve is the *strictly* trigonal (3,4) curve

$$C: \quad y^3 = x^4 + \lambda_3 x^3 + \cdots + \lambda_0,$$

which has genus 3. Now all functions are functions of $\mathbf{u} = (u_1, u_2, u_3)$. The $\wp$ functions satisfy a number of PDEs, the main ones of interest here are the generalisations of the genus 1 ODE

$$(\wp_{111})^2 = 4\wp_{11}^3 - g_2\wp_{11} - g_3.$$

In the genus 3 trigonal case we have 55 equations of this type

$$\begin{aligned}
\wp_{333}^2 &= 4\wp_{33}^3 + \wp_{23}^2 + 4\wp_{13} - 4\wp_{33}\wp_{22},\\
\wp_{233}\wp_{333} &= 4\wp_{23}\wp_{33}^2 - \wp_{22}\wp_{23} + 2\wp_{33}^2\lambda_3 - 2\wp_{12},\\
\wp_{233}^2 &= 4\wp_{33}\wp_{23}^2 + 4\wp_{33}\lambda_3\wp_{23} + \wp_{22}^2 + 4\wp_{33}\lambda_2 - \tfrac{4}{3}Q_{1333},\\
\ldots &= \ldots
\end{aligned}$$

where $Q_{1333} = \wp_{1333} - 6\wp_{33}\wp_{13}$.

# Kummer relations for the (3,4) curve

We can define weights in the (3,4) case in a similar way to the genus 2 case. We have 55 equations of the form $\wp_{ijk}\wp_{\ell mn} = \ldots$, starting from $\wp_{333}^2 = \ldots$ at wt. -6 and going down to $\wp_{111}^2 = \ldots$ at wt. -30. We now treat these as algebraic equations.

By eliminating the 3rd derivatives $\wp_{ijk}$ by a simple algebraic identity, we can find a large number of *Kummer relations* (KR) in the form

$$(\wp_{ijk}\wp_{\ell mn}) \cdot (\wp_{opq}\wp_{rst}) - (\wp_{ijk}\wp_{rst}) \cdot (\wp_{opq}\wp_{\ell mn}) = 0,$$

which can be written in terms of the *seven* variables $\wp_{11}, Q_{1333}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}$, each with a pole of order two.

We can grade the KRs by weight, from

$$K_{14} \equiv \wp_{333}^2 \cdot \wp_{233}^2 - (\wp_{233}\wp_{333})^2 = 0 \qquad [-14]$$

to

$$K_{54} \equiv \wp_{111}^2 \cdot \wp_{112}^2 - (\wp_{111}\wp_{112})^2 = 0 \qquad [-54]$$

Note we may have two or more KRs with the same weight.

# Kummer variety, (3,4) curve

More concise notation: $\wp_{11}, Q_{1333}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}$, become $X_{10}, X_8, X_7, X_6, X_4, X_3, X_2$.

In the (3,4) case, a surprising result is that the set of all KRs described above does *not* contain all the polynomial equations involving these underlying variables.

Calculations show an *extra* equation, which we will call $K_{12}$. It is quartic and of weight -12

$$
\begin{aligned}
K_{12} \equiv & -2\lambda_0 - 4X_2X_3X_7 + \lambda_3^2X_2^3 - 4\lambda_2X_2^3 - X_3^2\lambda_2 - 2X_6\lambda_2 - 8X_6X_2^3 \\
& - 4X_6X_3^2 + 4X_2X_4\lambda_2 + 6X_2X_4X_6 - 2X_6^2 - X_4X_8 - X_2^2X_4^2 \\
& + 2X_4X_2X_3^2 + 3X_4X_2\lambda_3X_3 + X_4^3 - 3X_2\lambda_3X_7 - 3X_3X_6\lambda_3 - \lambda_3X_3^3 \\
& - X_3^4 + \tfrac{4}{3}X_2^2X_8 \ - \lambda_1X_3 + 2X_2X_{10} = 0,
\end{aligned}
$$

On weight grounds this *cannot* belong to the set of KRs.

Where does this extra relation come from? Does it have any deeper significance?

# Gröbner Basis for the Kummer variety, (3,4) case

How many of the *825* KRs are independent?

We use *Gröbner Bases* (GB) to answer this question. Start from $K_{12}$ and keep adding the next lower weight KR. At each stage test the new equation using GB to see if it is independent.

We can use this algorithm to build up a list of KRs together with the corresponding GB. We find we need only *seven* relations in total, $K_{12}, K_{14}, K_{15}, K_{16a}, K_{16b}, K_{17}, K_{18}$. This set generates the Kummer variety in $(3, 4)$.

The use of weights and GB is also crucial in constructing the Coble quartic hypersurface associated with the genus three trigonal curve, restricting our calculation to a *finite* number of terms.

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The use of weights and GB is also crucial in constructing the Coble quartic hypersurface associated with the genus three trigonal curve, restricting our calculation to a *finite* number of terms.

# Coble quartic, $g = 3$ trigonal case

In genus 3, Coble theory tells us there is a unique quartic in $\mathbb{P}(V_2)$. The quartic is singular on the Kummer variety. We can generate it by using the Gröbner Basis. The partial derivatives of the quartic with respect to the $X_i$ give the cubic KRs, and with respect to the $\lambda_i$ give the quartic KRs. The net result is the unique weight 24 quartic

$$
\begin{aligned}
&-(4\,\lambda_0-\lambda_3\lambda_1)X_2{}^2X_8+3\,(2\,\lambda_0-2\,\lambda_3\lambda_1+\lambda_2{}^2)X_6{}^2+3\,(2\,\lambda_2-\lambda_3{}^2)X_6{}^3\\
&+3\,(4\,\lambda_0\lambda_2-\lambda_1{}^2-\lambda_0\lambda_3{}^2)X_2{}^3-3\,X_{10}X_4X_3X_7+3\,X_4{}^2X_6X_{10}+3\,\lambda_0X_3{}^4\\
&+3\,X_2\lambda_3X_7X_6{}^2+2\,X_7X_8X_3X_6-6\,\lambda_0X_3{}^2X_2X_4-3\,X_2{}^2\lambda_1X_4X_7\\
&+6\,X_6\lambda_2X_2X_{10}-3\,\lambda_1X_3X_2X_{10}+X_4X_8X_2X_{10}-3\,X_2{}^2\lambda_3X_7X_{10}\\
&+3\,X_2\lambda_1X_4X_3X_6-6\,X_6\lambda_2X_7X_2X_3+3\,X_3X_6\lambda_3X_2X_{10}+3\,X_6\lambda_1X_2X_4\lambda_3\\
&-9\,\lambda_0X_3X_2X_4\lambda_3-6\,X_2X_{10}X_6{}^2-X_4X_8X_6{}^2-3\,X_3X_6{}^3\lambda_3-3\,X_7{}^2X_4X_6\\
&+\quad\ldots\qquad\text{(74 terms in total).}
\end{aligned}
$$

In summary, the Coble polynomial encapsulate a large number of equations which give the algebraic structure of the PDEs associated with the genus 3 trigonal curve.

# Heat Equations

In genus *one*, we can rewrite the nonlinear ODE

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

as a nonlinear ODE in $u$ for $\sigma$

$$\begin{aligned}\sigma'''(u)^2\sigma(u)^2 - \sigma'(u)\sigma'''(u)(3\sigma''(u)\sigma(u) - 2\sigma'(u)^2) + 4\sigma''(u)^3\sigma(u)\\ + g_2\sigma(u)^2(\sigma'(u)^2 - \sigma(u)\sigma''(u)) - 3\sigma''(u)^2\sigma'(u)^2 + g_3\sigma(u)^4 = 0\end{aligned}$$

However, if we allow partial differentiation with respect to the curve moduli, $\sigma(u)$ satisfies two *linear* PDEs

$$\begin{aligned}L_1(\sigma) &\equiv \left(4g_2\frac{\partial}{\partial g_2} + 6g_3\frac{\partial}{\partial g_3}\right)\sigma = \left(u\frac{\partial}{\partial u} - 1\right)\sigma,\\ L_2(\sigma) &\equiv \left(6g_3\frac{\partial}{\partial g_2} + \frac{1}{3}g_2^2\frac{\partial}{\partial g_3}\right)\sigma = \left(\frac{1}{24}\frac{\partial^2}{\partial u^2} + \frac{g_2}{12}u^2\right)\sigma,\end{aligned}$$

(Abramowitz and Stegun).

# Recurrence relation, $g = 1$

We can get a recurrence relation for the coefficients of the sigma expansion in genus 1 in the form

$$\sigma(u) = u \sum_{n,m \geq 0} a_{n,m} (g_2 u^4)^n (g_3 u^6)^m$$

with $a_{0,0} = 1$ and $a_{n,m} = 0$ if $n < 0$ or $m < 0$. The first PDE above is identically satisfied, but the second gives, after a little algebra, the linear recurrence relation

$$a_{n,m} = \frac{1}{24(1+4n+6m)(3m+2n)} \times \\ \times \left[ 8(m+1)a_{n-2,m+1} - a_{n-1,m} + 144(n+1)a_{n+1,m-1} \right]$$

Note this does *not* depend on $g_2$ and $g_3$!
Buchstaber and Leykin developed a general theory for higher genus, but only gave worked examples in genus 2.

# Sigma function theory in genus 3

We work with a more general form of the curve, in "Weierstrass form"

$$y^3 + (\mu_8 + \mu_5 x + \mu_2 x^2)y = x^4 + \mu_6 x^2 + \mu_9 x + \mu_{12}.$$

We can derive an expansion for the $\sigma$ function expansion by complicated calculations using the nonlinear PDEs for the $\wp$ functions

$$\sigma(u_1, u_2, u_3) = C_5(u_1, u_2, u_3) + C_7(u_1, u_2, u_3) + C_9(u_1, u_2, u_3) + \cdots ,$$

where

$$\begin{aligned}
C_5 &= u_1 - u_3\, u_2{}^2 + \tfrac{1}{20}\, u_3{}^5, \\
C_7 &= -\tfrac{3}{504}\mu_2 u_3{}^7 + \tfrac{1}{6}\, \mu_2 u_3{}^3 u_2{}^2, \\
C_9 &= -\tfrac{9}{25920}\mu_2^2 u_3{}^9 - \tfrac{1}{120}\mu_2^2 u_3{}^5 u_2{}^2 - \tfrac{1}{12}\mu_2^2 u_3 u_2{}^4, \\
C_{10} &= -\tfrac{1}{1680}\mu_5 u_3{}^8 u_2 - \tfrac{1}{14}\mu_5 u_2{}^5 + \tfrac{1}{6}\, \mu_5 u_3{}^3 u_2 u_1, \\
C_{11} &= \ldots ,
\end{aligned}$$

# Linear equations for $\sigma(u)$ in genus 3

We can get a linear set of PDEs for $\sigma$ in genus 3, following the Buchstaber & Leykin approach. There are now six ($2g$) $L_i$ operators (vector fields tangent to the discriminant)

$$\begin{aligned}
L_1 &= 2\,\mu_2\frac{\partial}{\partial\mu_2} + 5\,\mu_5\frac{\partial}{\partial\mu_5} + 6\,\mu_6\frac{\partial}{\partial\mu_6} + 8\,\mu_8\frac{\partial}{\partial\mu_8} + 9\,\mu_9\frac{\partial}{\partial\mu_9} + 12\,\mu_{12}\frac{\partial}{\partial\mu_{12}},\\
L_2 &= 5\,\mu_5\frac{\partial}{\partial\mu_2} - \tfrac{1}{6}\,\left({\mu_2}^4 + 24\,\mu_2\mu_6 - 48\,\mu_8\right)\frac{\partial}{\partial\mu_5} + \tfrac{1}{2}\,\left(\mu_2^2\mu_5 + 18\,\mu_9\right)\frac{\partial}{\partial\mu_6}\\
&\quad - \tfrac{1}{12}\left(\mu_2^3\mu_5 + 18\,\mu_5\mu_6 + 6\,\mu_2\mu_9\right)\frac{\partial}{\partial\mu_8} - \tfrac{1}{6}\left(\mu_2^3\mu_6 - \mu_2\mu_5^2 + 18\,\mu_6^2 - 2\,\mu_2^2\mu_8\right.\\
&\quad \left.- 72\,\mu_{12}\right)\frac{\partial}{\partial\mu_9} - \tfrac{1}{12}\,\left(\mu_2^3\mu_9 - 2\,\mu_2\mu_5\mu_8 + 18\,\mu_6\mu_9\right)\frac{\partial}{\partial\mu_{12}},\\
L_3 &= \ldots
\end{aligned}$$

and $\sigma$ satisfies six linear PDEs, the first being

$$L_1\,\sigma = \left[5\,u_1\frac{\partial}{\partial u_1} + 2\,u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3} - 5\right]\sigma,$$

# Linear PDEs for $\sigma$

$$
\begin{aligned}
L_2\,\sigma = \Big[ & 2\,u_2\frac{\partial}{\partial u_1} - \tfrac{1}{12}\left(18\mu_6+\mu_2^3\right)u_1\frac{\partial}{\partial u_2} + \tfrac{1}{2}\,\mu_2 u_3\frac{\partial}{\partial u_2} - \tfrac{1}{6}\,\mu_5\mu_2 u_1\frac{\partial}{\partial u_3} \\
& -\tfrac{1}{3}\mu_2^2 u_2\frac{\partial}{\partial u_3} + \frac{\partial^2}{\partial u_3 \partial u_2} - \tfrac{1}{24}\left(\mu_2^4\mu_5 + 4\mu_2^2\mu_9 + 20\mu_2\mu_5\mu_6 - 60\,\mu_8\mu_5\right)u_1^2 \\
& -\mu_8\mu_2 u_1 u_2 + \mu_9 u_1 u_3 + \tfrac{1}{3}\,\mu_5\mu_2 {u_2}^2 - \tfrac{1}{6}\left({\mu_2}^3 + 6\,\mu_6\right)u_2 u_3 - \tfrac{1}{2}\,\mu_5 {u_3}^2\Big]\,\sigma \\
L_3\,\sigma = \ldots &
\end{aligned}
$$

Following BL05, in the (3,4) case we define

$$
\begin{aligned}
\sigma(u_1,u_2,u_3) = u_3^5 \sum_{\ell,m,n,\dots\geq 0} & a_{l,m,n,o,p,q,r,t}\left(\frac{u_1}{u_3^5}\right)^{\ell}\left(\frac{u_2}{{u_3}^2}\right)^{m}\left(\mu_2 {u_3}^2\right)^{n}\times \\
& \times\left(\mu_5 {u_3}^5\right)^{o}\left(\mu_6 {u_3}^6\right)^{p}\left(\mu_8 {u_3}^8\right)^{q}\left(\mu_9 {u_3}^9\right)^{r}\left(\mu_{12} {u_3}^{12}\right)^{t}
\end{aligned}
$$

# $\sigma$ expansion

If we define

$$k = 5 - 5\ell - 2m + 2n + 5o + 6p + 8q + 9r + 12t,$$

we can rewrite the previous expression as

$$\sigma = \sum a_{\ell,m,n,o,p,q,r,t}\, \mu_2^n\, \mu_5^o\, \mu_6^p\, \mu_8^q\, \mu_9^r\, \mu_{12}^t\, u_1^\ell u_2^m u_3^k,$$

where all the integer indices satisfy $k, \ell, m, n, o, p, q, r, t \geq 0$. Inserting this into the $L_i$ equations, the first is identically true, the remaining five give various recurrence relations. For example, from $L_6$ we get a relation connecting 42 terms. Solving this for the highest weight coefficient gives

$$\begin{aligned} P_6: \quad a_{\ell,m,n,o,p,q,r,t} =& \frac{1}{\ell(\ell-1)}\Big( -\tfrac{1}{3}(k+1)(2+k)a_{\ell-2,m,n,o,p,q-1,r,t} \\ & - \tfrac{1}{2}(m+1)a_{\ell-2,m+1,n,o,p,q,r-1,t} + \text{39 other terms}\Big). \end{aligned}$$

Clearly we can't use this when $\ell = 0, 1$.

# General recurrence relation for sigma, $g = 3$

With some care we can eventually construct a series of options for the *linear* relations which will always eventually give higher weight terms in the $\sigma$ expansion in terms of lower weight terms. We set $a_{1,0,\ldots} = 1$ as an overall normalisation. The prescription is

$$
\begin{aligned}
a_{\ell,m,n,o,p,q,r,t} &= 0 \quad \text{if } \min(\ell,m,n,o,p,q,r,t) < 0, \\
&= \text{rhs}(P_6) \quad \text{if } \ell > 1, \\
&= \text{rhs}(P_5) \quad \text{if } \ell > 0, m > 0, \\
&= \text{rhs}(P_4) \quad \text{if } \ell > 0, k > 0, \\
&= \text{rhs}(T_2) \quad \text{if } \ell = 0, m \neq 0 \text{ and } m \neq (k+1), \\
&= \text{rhs}(T_3^{(0)}) \quad \text{if } \ell = m = 0, \\
&= \text{rhs}(P_2^{(0)}) \quad \text{if } \ell = 0, \text{ and } m = (k+1),
\end{aligned}
$$

where the $P_i$ follow directly from the $L_i$; $T_2$, $T_3$ are linear combinations of $P_2, P_3$, and the 0 superscript represents some special choices of $\ell, m$ and appropriate terms. This algorithm, although complicated, is very fast and only limited by memory requirements.

# Some further reading


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