Extended KP hierarchies with self-consistent sources and Binary Darboux Transformations

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Abstract. Extensions of the k-constrained KP hierarchy (k-cKPH) are presented. These hierarchies cover (2+1)-dimensional extensions of the k-cKPH and include matrix Davey-Stewartson equation, generalization of the N-wave system and extended KP equation with self-consistent sources. Binary Darboux Transformations are applied to construct solutions of the presented hierarchies.

Keywords: KP hierarchy, symmetry constraints, Binary Darboux Transformation, KP equation with self-consistent sources
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INTRODUCTION

An important place in the theory of integrable systems is occupied by the algebraic constructions of the well-known Kyoto group [1], which are called the Sato theory. Family of equations of Kadomtsev-Petviashvili type (KP hierarchy) occupy an important role in these investigations.

One of known generalizations of the KP hierarchy arise as a result of k-symmetry constraints (so-called k-cKP hierarchy) [2, 3, 4], which is related with the KP equation with self-consistent sources (KPSCS) [5].

Multicomponent k-constraints of the KP hierarchy were proposed and and investigated in [6, 7]. This extension of k-cKP hierarchy contains vector (multicomponent) generalizations of the nonlinear Schrödinger equation, the Yajima-Oikawa system, a generalization of the Boussinesq equation, and the KPSCS.

In [8] (2+1)-dimensional extensions of the k-cKP hierarchy ((2+1)-dimensional k-cKP hierarchy) were introduced and solution generating methods for them were investigated. These hierarchies were also considered recently in [9].

Group-theoretical and Lie-algebraic scheme of the Hamiltonian analysis of these and some other integrable hierarchies (including BDk-cKP hierarchy [10]) was investigated in [11, 12]. Analytical theory of integrability of Hamiltonian systems was investigated in [13].

In the following section we present new extensions of the (2+1)-dimensional k-cKP hierarchy. In contrast to the above-mentioned hierarchies the latter, in particular, include the equation, which contains two types of KP equation with self-consistent sources as subcases. We also present a solution generating method for the proposed hierarchies via Binary Darboux Transformations.

A NEW (2+1)-DIMENSIONAL GENERALIZATION OF THE K-CONSTRAINED KP HIERARCHY

Introduce the following pair of integro-differential operators L_k and M_n:

$$\begin{aligned} L_k &= \beta_k \partial_{\tau_k} - B_k - \mathbf{q}\mathscr{M}_0 D^{-1}\mathbf{r}^\top,\ B_k = \textstyle\sum_{j=0}^{k} u_j D^j, u_j = u_j(x,\tau_k,t_n),\ \beta_k \in C, \\ M_n &= \alpha_n \partial_{t_n} - A_n - \tilde{\mathbf{q}}\tilde{\mathscr{M}}_0 D^{-1}\tilde{\mathbf{r}}^\top,\ A_n = \textstyle\sum_{i=0}^{n} v_i D^i, v_i = v_i(x,\tau_k,t_n), \alpha_n \in C, \end{aligned} \tag{1}$$

where u_j and v_i are matrix-valued functions of dimension $N \times N$; $\mathbf{q}$ and $\mathbf{r}$ are matrix-valued functions of dimension $N \times m$; $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{r}}$ are matrix-valued functions with dimension $N \times \tilde{m}$. $\mathscr{M}_0$ and $\tilde{\mathscr{M}}_0$ are constant matrices with dimensions $m \times m$ and $\tilde{m} \times \tilde{m}$ respectively. We shall also assume that functions $\mathbf{q}$, $\mathbf{r}$, $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{r}}$ satisfy equations:

$$L_k\{\tilde{\mathbf{q}}\} = \tilde{\mathbf{q}}\Lambda_{\tilde{\mathbf{q}}}, \quad L_k^\tau\{\tilde{\mathbf{r}}\} = \tilde{\mathbf{r}}\Lambda_{\tilde{\mathbf{r}}}, \quad M_n\{\mathbf{q}\} = \mathbf{q}\Lambda_{\mathbf{q}},\ M_n^\tau\{\mathbf{r}\} = \mathbf{r}\Lambda_{\mathbf{r}}. \tag{2}$$

The latter pair of operators satisfies the following proposition:
Proposition. *The Lax equation $[L_k, M_n] = 0$ holds in case the following equations are satisfied:*

$$\begin{aligned} &[L_k, M_n]_{\geq 0} = 0, \qquad L_k\{\tilde{\mathbf{q}}\} = \tilde{\mathbf{q}}\Lambda_{\tilde{\mathbf{q}}}, \quad L_k^\tau\{\tilde{\mathbf{r}}\} = \tilde{\mathbf{r}}\Lambda_{\tilde{\mathbf{r}}}, \\ &M_n\{\mathbf{q}\} = \mathbf{q}\Lambda_{\mathbf{q}},\ M_n^\tau\{\mathbf{r}\} = \mathbf{r}\Lambda_{\mathbf{r}}, \end{aligned} \tag{3}$$

where $\Lambda_{\mathbf{q}}$, $\Lambda_{\mathbf{r}}$, $\Lambda_{\tilde{\mathbf{q}}}$, $\Lambda_{\tilde{\mathbf{r}}}$ are constant matrices with dimensions ($m \times m$) and ($\tilde{m} \times \tilde{m}$) that satisfy equations: $\Lambda_{\tilde{\mathbf{q}}}\tilde{\mathscr{M}}_0 - \tilde{\mathscr{M}}_0\Lambda_{\tilde{\mathbf{r}}}^\top = 0$, $\Lambda_{\mathbf{q}}\mathscr{M}_0 - \mathscr{M}_0\Lambda_{\mathbf{r}}^\top = 0$.

Consider some nonlinear systems that hierarchies given by (1)-(3) contain. For simplicity we set $\Lambda_{\mathbf{q}} = \Lambda_{\mathbf{r}} = 0$, $\Lambda_{\tilde{\mathbf{q}}} = \Lambda_{\tilde{\mathbf{r}}} = 0$.

1. $k = 1, n = 1$. We shall use the following notation $\beta := \beta_1$, $\alpha := \alpha_1$, $\tau := \tau_1$, $t := t_1$.

   $$L_1 = \beta\partial_\tau - JD + [J,Q] - \mathbf{q}\mathscr{M}_0 D^{-1}\mathbf{r}^\top, \quad M_1 = \alpha\partial_t - \tilde{J}D + [\tilde{J},Q] - \tilde{\mathbf{q}}\tilde{\mathscr{M}}_0 D^{-1}\tilde{\mathbf{r}}^\top.$$

   Under the Hermitian conjugation reduction $\tilde{\mathbf{r}} = \bar{\tilde{\mathbf{q}}}$, $\mathscr{M}_0 = \mathscr{M}_0^*$, $\tilde{\mathscr{M}}_0 = \tilde{\mathscr{M}}_0^*$, $\mathbf{r} = \bar{\mathbf{q}}$, $Q = -Q^*$, $\alpha, \beta \in R$, $J = J^*$, $\tilde{J} = \tilde{J}^*$ Lax equation $[L_1, M_1] = 0$ is equivalent to the following extension of the N-wave system.

   $$\begin{aligned} &\beta[\tilde{J}, Q_\tau] - \alpha[J, Q_t] + JQ_x\tilde{J} - \tilde{J}Q_xJ + [[J,Q],[\tilde{J},Q]] + \\ &+[J, \tilde{\mathbf{q}}\tilde{\mathscr{M}}_0\tilde{\mathbf{q}}^*] - [\tilde{\mathbf{q}}\tilde{\mathscr{M}}_0\tilde{\mathbf{q}}^*, \tilde{J}] = 0, \quad S_{1,x} = \mathbf{q}^*\tilde{\mathbf{q}}, \\ &\beta\tilde{\mathbf{q}}_\tau - J\tilde{\mathbf{q}}_x + u\tilde{\mathbf{q}} - \mathbf{q}\mathscr{M}_0 S_1 = 0, \qquad \alpha\mathbf{q}_t - \tilde{J}\mathbf{q}_x + v\mathbf{q} - \tilde{\mathbf{q}}\tilde{\mathscr{M}}_0 S_1^* = 0. \end{aligned}$$

   The latter also generalizes the system of four waves, which was investigated via the Inverse Scattering Method in [14].
2. $k = 1, n = 2$.

   $$L_1 = \beta_1\partial_{\tau_1} - \mathbf{q}\mathscr{M}_0 D^{-1}\mathbf{r}^\top, \qquad M_2 = \alpha_2\partial_{t_2} - D^2 + v - \tilde{\mathbf{q}}\tilde{\mathscr{M}}_0 D^{-1}\tilde{\mathbf{r}}^\top.$$

   Lax equation $[L_1, M_2] = 0$ is equivalent to the following generalization of the DS-III system:

   $$\begin{aligned} &\beta_1\tilde{\mathbf{q}}_{\tau_1} + u\tilde{\mathbf{q}} = \mathbf{q}\mathscr{M}_0 S_1, \qquad S_{1,x} = \mathbf{r}^\top\tilde{\mathbf{q}}, \qquad \beta_1\tilde{\mathbf{r}}_{\tau_1}^\top - \tilde{\mathbf{r}}^\top u = S_2\mathscr{M}_0\mathbf{r}^\top, \\ &\alpha_2\mathbf{q}_{t_2} - c\mathbf{q}_{xx} + v\mathbf{q} = \tilde{\mathbf{q}}\tilde{\mathscr{M}}_0 S_2, \qquad S_{2,x} = \tilde{\mathbf{r}}^\top\mathbf{q}, \qquad \alpha_2\mathbf{r}_{t_2}^\top + \mathbf{r}_{xx}^\top - \mathbf{r}^\top v = S_1\tilde{\mathscr{M}}_0\tilde{\mathbf{r}}^\top, \\ &\beta_1 v_{\tau_1} = 2(\mathbf{q}\mathscr{M}_0\mathbf{r}^\top)_x. \end{aligned}$$

   In case we set $\tilde{\mathbf{q}} = 0$, $\tilde{\mathbf{r}} = 0$ the DS-III system is recovered.

3. $k=3, n=2$.

$$L_3=\beta_3\partial_{\tau_3}-D^3+wD+u-\mathbf{q}\mathscr{M}_0 D^{-1}\mathbf{r}^\top, \qquad M_2=\alpha_2\partial_{t_2}-c_2D^2+v-\tilde{\mathbf{q}}\tilde{\mathscr{M}}_0 D^{-1}\tilde{\mathbf{r}}^\top.$$

In the scalar case ($N=1$) under the Hermitian conjugation reduction: $\alpha_2\in iR$, $\mathbf{r}=\bar{\mathbf{q}}$, $\mathscr{M}_0=\mathscr{M}_0^*$ ($M_2=M_2^*$) and $\beta_3\in R$, $\tilde{\mathscr{M}}_0=-\tilde{\mathscr{M}}_0^*$, $\tilde{\mathbf{r}}=\bar{\tilde{\mathbf{q}}}$, $w=w^*$, $w_x^*=u+u^*$, $v=v^*$ ($L_3=-L_3^*$) Lax equation $[L_3,M_2]=0$ is equivalent to the system:

$$\begin{array}{l}\left(\beta_3 v_{\tau_3}-\frac{1}{4}v_{xxx}+\frac{3}{2}vv_x\right)_x-3\alpha^2 v_{t_2t_2}+\\ +\frac{3}{2}\left(\tilde{\mathbf{q}}_{xx}\tilde{\mathscr{M}}_0\tilde{\mathbf{q}}^*-\tilde{\mathbf{q}}\tilde{\mathscr{M}}_0\tilde{\mathbf{q}}_{xx}^*+\alpha(\tilde{\mathbf{q}}\tilde{\mathscr{M}}_0\tilde{\mathbf{q}}^*)_t\right)_x-2\left(\mathbf{q}\mathscr{M}_0\mathbf{q}^*\right)_{xx}=0,\\ \beta_3\tilde{\mathbf{q}}_{\tau_3}-\tilde{\mathbf{q}}_{xxx}+\frac{3}{2}v\tilde{\mathbf{q}}_x+u\tilde{\mathbf{q}}-\mathbf{q}\mathscr{M}_0S_1=0, \quad S_{1,x}=\mathbf{q}^*\tilde{\mathbf{q}}\\ \alpha_2\mathbf{q}_{t_2}-\mathbf{q}_{xx}+v\mathbf{q}-\tilde{\mathbf{q}}\tilde{\mathscr{M}}_0S_1^*=0.\end{array}$$

This system generalizes the KP equation with self-consistent sources (KPSCS). Case $\tilde{\mathscr{M}}_0=0$, $\tilde{\mathbf{q}}=0$ corresponds to the KPSCS of the first type. If $\mathscr{M}_0=0$, $\mathbf{q}=0$ we recover KPSCS of the second type.

SOLUTION GENERATING METHODS VIA BINARY DARBOUX TRANSFORMATIONS

Let $N\times K$-matrix functions φ and ψ be solutions of linear problems:

$$L_k\{\varphi\}=\varphi\Lambda,\ L_k^\tau\{\psi\}=\psi\tilde{\Lambda},\ \Lambda,\tilde{\Lambda}\in Mat_{K\times K}(C). \tag{4}$$

Introduce binary Darboux transformation (BDT) in the following way:

$$W=I-\varphi\left(C+D^{-1}\{\psi^\top\varphi\}\right)^{-1}D^{-1}\psi^\top, \tag{5}$$

where C is a $K\times K$-constant nondegenerate matrix. The following theorem holds:

Theorem. *Let $N\times K$-matrix functions φ and ψ satisfy equations:*

$$\begin{array}{c}L_k\{\varphi\}=\varphi\Lambda_k,\ L_k^\tau\{\psi\}=\psi\tilde{\Lambda}_k,\ \Lambda_k,\tilde{\Lambda}_k\in Mat_{K\times K}(C),\\ M_n\{\varphi\}=\varphi\Lambda_n,\ M_k^\tau\{\psi\}=\psi\tilde{\Lambda}_n,\ \Lambda_n,\tilde{\Lambda}_n\in Mat_{K\times K}(C)\end{array} \tag{6}$$

with operators L_k and M_n (1) satisfying $[L_k,M_n]=0$. Then transformed operators $\hat{L}_k$ and $\hat{M}_n$ satisfy the Lax equation $[\hat{L}_k,\hat{M}_n]=0$ and have the form:

$$\begin{array}{l}\hat{L}_k:=WL_kW^{-1}=\beta_k\partial_{\tau_k}-\hat{B}_k-\hat{\mathbf{q}}\mathscr{M}_kD^{-1}\hat{\mathbf{r}}^\top+\Phi\mathscr{M}_1D^{-1}\Psi^\top,\ \hat{B}_k=\sum_{j=0}^k\hat{u}_jD^j,\\ \hat{M}_n:=WM_nW^{-1}=\alpha_n\partial_{t_n}-\hat{A}_n-\hat{\tilde{\mathbf{q}}}\tilde{\mathscr{M}}_0D^{-1}\hat{\tilde{\mathbf{r}}}^\top+\Phi\mathscr{M}_nD^{-1}\Psi^\top,\ \hat{A}_n=\sum_{i=0}^n\hat{v}_iD^i.\end{array} \tag{7}$$

where

$$\begin{array}{l}\mathscr{M}_k=C\Lambda_k-\tilde{\Lambda}_k^\top C,\ \mathscr{M}_n=C\Lambda_n-\tilde{\Lambda}_n^\top C,\ \Phi=\varphi\Delta^{-1},\ \Psi=\psi\Delta^{-1,\top},\\ \Delta=C+D^{-1}\{\psi^\top\varphi\},\ \hat{\mathbf{q}}=W\{\mathbf{q}\},\ \hat{\mathbf{r}}=W^{-1,\tau}\{\mathbf{r}\},\ \hat{\tilde{\mathbf{q}}}=W\{\tilde{\mathbf{q}}\},\ \hat{\tilde{\mathbf{r}}}=W^{-1,\tau}\{\tilde{\mathbf{r}}\}.\end{array} \tag{8}$$

$\hat{u}_j$, $\hat{v}_i$ are $N\times N$-matrix coefficients depending on functions φ, ψ and u_j, v_i. In particular,

$$\begin{array}{l}\hat{u}_k=u_k,\ \hat{u}_{k-1}=u_{k-1}+\left[u_k,\varphi\left(C+D^{-1}\{\psi^\top\varphi\}\right)^{-1}\psi^\top\right],\\ \hat{v}_n=v_n,\ \hat{v}_{n-1}=v_{n-1}+\left[v_n,\varphi\left(C+D^{-1}\{\psi^\top\varphi\}\right)^{-1}\psi^\top\right].\end{array} \tag{9}$$

CONCLUSIONS

In this work we propose extensions of the (2+1)-dimensional k-constrained KP hierarchies [8, 10] and apply BDTs to generate families of solutions for the corresponding nonlinear integrable systems.

It is known that for some equations with self-consistent sources classes of solutions were recently generated via the Inverse Scattering Method. It concerns, in particular, matrix KdV equation with self-consistent sources [15]. One of the problems of future interest consists in extending of those results to the equations that belong the constructed hierarchies. Another important question for future investigation concerns construction of analogues of rogue wave solutions of NLS, and KP equations [16, 17].

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