KP hierarchy for a cyclic quiver

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Joint work (in preparation) with Alexey Silantyev.
Related works:

- G. Wilson: Collisions of Calogero–Moser particles and adelic Grassmannian. Invent. Math. (1998)
- Berest–Wilson; Kapustin–Kuznetsov–Orlov; Baranovsky–Ginzburg–Kuznetsov; Berest–C.; F. Eshmatov
- Bakalov–Horozov–Yakimov (1997); Rothstein (1998)

One-dimensional Dunkl operator for $\Gamma = \mathbb{Z}_m$

$\Gamma = \{1, s, s^2, \ldots, s^{m-1}\}$ - cyclic group of order m

The group algebra $\mathbb{C}\Gamma$ is spanned by m idempotents

$$\epsilon_i = \frac{1}{m} \sum_{j=0}^{m-1} \mu^{-ij} s^j \, , \quad \mu = \exp \frac{2\pi\sqrt{-1}}{m}$$

E.g. $\epsilon_0 = \frac{1}{m} \sum_{j=0}^{m} s^j$ (symmetrizer).

Dunkl operator is the following element of $\mathcal{D}(x, \partial) * \Gamma$:

$$y = \partial - \frac{1}{x} \sum_{i=0}^{m-1} k_i \epsilon_i \qquad (k_i \in \mathbb{C} \text{ are parameters})$$

Inside $\mathcal{D}(x, \partial_x) * \Gamma$ we have the **Cherednik algebra** $H_k(\mathbb{Z}_m)$, generated by x, y and s, with relations:

$$yx - xy = \sum_i \lambda_i \epsilon_i \, , \quad sx = \mu^{-1} xs \, , \quad sy = \mu ys \, .$$

Here $\lambda_i = k_{i-1} - k_i + 1/m$, so $\lambda_0 + \cdots + \lambda_{m-1} = 1$.

The generalised KP hierarchy

The usual KP hierarchy can be defined using pesudo-differential operators. We consider instead the following formal expressions:

$$A = \sum_{n \geq -N} a_n y^{-n}, \quad a_n \in \mathbb{C}(x) * \Gamma \quad (\Gamma = \mathbb{Z}_m)$$

One can define an associative multiplication of such expressions in a natural way. (**NB**: multiplication depends on the parameters λ_i.)
Now we put

$$L = y + \sum_{n \geq 0} f_n y^{-n}, \qquad f_n \in \mathbb{C}(x; t_1, t_2, \dots) * \Gamma$$

$$\frac{\partial L}{\partial t_j} = [(L^{mj})_+, L] \quad (j = 1, 2, \dots)$$

Here A_+, A_- are defined in the usual way. This is a compatible system of equations on the coefficients of L. Note that due to the non-local nature of the Dunkl operator y, these equations are not purely differential.
For $m = 1$: $\Gamma = \{1\}$, $y = \partial \Rightarrow$ the usual KP hierarchy.

Preprojective algebras

Consider a quiver Q, an oriented graph with the set of vertices I and arrows Q. Write $\overline{Q}$ for the double quiver, obtained by adding a reverse arrow $a^* : j \to i$ for every arrow $a : i \to j$ in Q. Let $\mathbb{C}\overline{Q}$ be the path algebra of $\overline{Q}$; it has basis the paths in $\overline{Q}$, including a trivial path e_i for each vertex.

For $\lambda \in \mathbb{C}^I$ consider the preprojective algebra (Crawley-Boevey, Holland):

$$\Pi^\lambda(Q) = \mathbb{C}\overline{Q}/(\sum_{a \in Q} [a, a^*] - \sum_{i \in I} \lambda_i e_i) \, .$$

Representations of $\overline{Q}$ of dimension $\alpha \in \mathbb{N}^I$:

$$\mathtt{Rep}(\overline{Q}, \alpha) = \bigoplus_{x : i \to j} \mathtt{Hom}(\mathbb{C}^{\alpha_i}, \mathbb{C}^{\alpha_j}) \oplus \mathtt{Hom}(\mathbb{C}^{\alpha_j}, \mathbb{C}^{\alpha_i})$$

We have naturally $\mathtt{Rep}(\Pi^\lambda(Q), \alpha) \subset \mathtt{Rep}(\overline{Q}, \alpha)$ as a subvariety. The isomorphism classes of representations correspond to the orbits under the action of the group $\mathtt{GL}(\alpha) = \prod_{i \in I} \mathtt{GL}(\alpha_i, \mathbb{C})$.

Cherednik algebra $H_k(\mathbb{Z}_m)$ as preprojective algebra

Let Q be a cyclic oriented quiver with the set of vertices $I = \mathbb{Z}/m\mathbb{Z} = \{0, 1, \ldots, m-1\}$ and m arrows $a_i : i \to i+1$. To double the quiver, we add to it the reverse arrows $a_i^* : i+1 \to i$. Choose m complex parameters λ_i, $i \in I$; the correponding preprojective algebra $\Pi^\lambda(Q)$ is the quotient of the path algebra of the doubled cyclic quiver by the relation

$$\sum_{i \in I} [a_i, a_i^*] - \sum_{i \in I} \lambda_i e_i = 0$$

This can be rewritten as m relations

$$a_{i-1} a_{i-1}^* - a_i^* a_i = \lambda_i e_i \,, \quad i \in \mathbb{Z}/m\mathbb{Z} \,.$$

If $m = 1$, then we have a one-loop quiver Q and Π^λ is isomorphic to $\mathbb{C}\langle a, a^* \rangle / \{aa^* - a^* a = \lambda\}$. For $m > 1$ there is an isomorphism between $\Pi^\lambda(Q)$ and Cherednik algebra $H_k(\mathbb{Z}_m)$, given by

$$a_i \mapsto \epsilon_i y \,, \quad a_i^* \mapsto \epsilon_i x \,, \quad e_i \mapsto \epsilon_i \,.$$

Calogero–Moser quiver Q_∞

Let Q be a cyclic oriented quiver on m vertices, Q_∞ be the quiver obtained from Q by adding a vertex ∞ and an arrow $v : \infty \to 0$, and $\Pi^\lambda(Q_\infty)$ be its preprojective algebra. A representation of $\Pi^\lambda(Q_\infty)$ is described by:

- a vector space $V_\infty \oplus V_0 \oplus \cdots \oplus V_{m-1}$;
- a collection of linear maps

$$X_i : V_i \to V_{i+1}\,,\; Y_i : V_{i+1} \to V_i\,,\; v : V_\infty \to V_0\,,\; w : V_0 \to V_\infty\,;$$

- relations

$$X_{i-1}Y_{i-1} - Y_iX_i + \delta_{i,0}vw = \lambda_i \mathtt{Id}_{V_i} \qquad (i \in \mathbb{Z}/m\mathbb{Z})$$
$$-wv = \lambda_\infty \mathtt{Id}_{V_\infty}$$

By adding these and taking traces, we get $\lambda \cdot \alpha = 0$, where $\alpha = (\dim V_\infty, \dim V_0, \ldots, \dim V_{m-1})$. We can always ensure $\lambda \cdot \alpha = 0$ by choosing λ_∞ appropriately.

For $m = 1$, $\lambda_0 = 1$ one gets the Wilson's space of quadruples (Y_0, X_0, v, w) with $Y_0X_0 - X_0Y_0 + vw = \mathtt{Id}$.

From quiver representations to solutions

Let us now restrict to representations with $\dim V_\infty = 1$. Put $X = X_0 + \cdots + X_{m-1}$ and $Y = Y_0 + \cdots + Y_{m-1}$; these are linear endomorphisms of $V := V_0 \oplus V_1 \oplus \cdots \oplus V_{m-1}$. Identifying V_∞ with $\mathbb{C}$, we can think of $v : V_\infty \to V_0 \subset V$ and $w : V_0 \to V_\infty$ as elements of V and its dual (i.e. extend w by zero outside of V_0). For $\mathbf{t} := (t_1, t_2, \dots)$ define

$$X(\mathbf{t}) = X - \sum_{j \geq 1} j t_j Y^{mj-1}$$

and put $M = 1 - \epsilon_0 w(X(\mathbf{t}) - x\mathtt{Id}_V)^{-1}(Y - y\mathtt{Id}_V)^{-1} v \epsilon_0$. A short calculation shows that

$$M^{-1} = 1 + \epsilon_0 w(Y - y\mathtt{Id}_V)^{-1}(X(\mathbf{t}) - x\mathtt{Id}_V)^{-1} v \epsilon_0 \,.$$

Theorem. For any representation of $\Pi^\lambda(Q_\infty)$ with $\dim V_\infty = 1$, the operator $L(\mathbf{t}) = M y M^{-1}$ is a solution of the generalised KP hierarchy.

Calogero–Moser varieties

Consider representations of $\Pi^\lambda = \Pi^\lambda(Q_\infty)$ of fixed dimension $\alpha = (\alpha_\infty, \alpha_0, \ldots, \alpha_{m-1}) \in \mathbb{N}^{m+1}$, assuming $\alpha_\infty = 1$ and $\lambda \cdot \alpha = 0$. Define the Calogero-Moser variety as the space of isomorphism classes of such representations, i.e.

$$C_{\lambda,\alpha} = \mathtt{Rep}(\Pi^\lambda, \alpha)//\mathtt{GL}(\alpha)\,.$$

Assuming that the parameters λ_i are generic (in a certain precise sense), one can use the work of Crawley–Boevey to show that:
(1) The space $C_{\lambda,\alpha}$ is non-empty iff α is a positive root of the Kac–Moody algebra associated to the quiver Q_∞.
(2) For any such root α, $C_{\lambda,\alpha}$ is a smooth conected affine algebraic variety of dimension

$$\dim_{\mathbb{C}} C_{\lambda,\alpha} = 2\alpha_0 - \sum_{i\in\mathbb{Z}/m\mathbb{Z}} (\alpha_i - \alpha_{i+1})^2\,.$$

E.g. for $\alpha = (1, n, n, \ldots, n)$ the Calogero–Moser variety is $2n$-dimensional.

Dynamics of poles

Suppose $\alpha = (1, n, \ldots, n)$, i.e. $\dim V_i = n$ for all $i = 0, \ldots, m-1$. Let $\lambda = (\lambda_\infty, \lambda_0, \ldots, \lambda_{m-1})$ where λ_i are generic and $\lambda \cdot \alpha = 0$. Then the CM variety $C_{\lambda,\alpha}$ is smooth of complex dimension $2n$. Each point of $C_{\lambda,\alpha}$ is represented by a pair of block matrices X, Y

$$X = X_0 + \cdots + X_{m-1}, Y = Y_0 + \cdots + Y_{m-1}$$

of size $mn \times mn$ and a pair v, w (vector/covector). Generically on $C_{\lambda,\alpha}$, the matrix X will have simple eigenvalues.
Recall $X(\mathbf{t}) = X - \sum_{j\geq 1} jt_j Y^{mj-1}$ and

$$M = 1 - \epsilon_0 w(X(\mathbf{t}) - x\mathtt{Id}_V)^{-1}(Y - y\mathtt{Id}_V)^{-1} v\epsilon_0$$

Clearly, the poles in x of $L(\mathbf{t}) = MyM^{-1}$ correspond to eigenvalues of $X(\mathbf{t})$.
Theorem. The spectrum of $X(\mathbf{t})$ is a union of n m-tuples of the form $\mu^i z_j(\mathbf{t})$ ($\mu^m = 1$), where the dynamics of $z_1, \ldots, z_n$ is governed by the rational Calogero–Moser system for the complex reflection group $G(n, 1, m) = S_n \wr \mathbb{Z}_m$.

Calogero–Moser system for $W = S_n \wr \mathbb{Z}_m$

For each complex reflection group W, the Calogero–Moser system consists of the Poisson-commuting Hamiltonians constructed using (classical) Dunkl operators. The group $W = \mathbb{Z}_m \wr S_n$ is a particular case; it can be described as the group of permutation n by n matrices whose the entries are arbitrary m-th roots of unity. The $m = 1$ case is the usual symmetric group; the $m = 2$ case gives the group of type B_n.

The group $W = S_n \wr \mathbb{Z}_m$ acts naturally in its reflection representation, $\mathbb{C}^n$. In this case we have n commuting Dunkl operators ∇_i $(i = 1, \ldots, n)$ and the commuting Hamiltonians are obtained as

$$H_j = \sum_{i=1}^{n} (\nabla_i)^{mj}\,, \qquad j = 1, 2, \ldots\,.$$

Then the dynamics of $(z_1, \ldots, z_n) \in \mathbb{C}^n$ with respect to the Hamiltonian flow corresponding to H_j is precisely the dynamics of the poles $z_i(\mathbf{t})$ of the above $L(\mathbf{t})$ with respect to t_j.

Relation to Darboux transformations of Bessel type

Rational solutions to KP hierarchy can be constructed using Darboux transformations starting from a differential operator with constant coefficients, $L_0 = p(\partial)$, and by factorizing it into a composition of non-trivial differential operators A, B:

$$L_0 = AB\,, \quad L_1 := BA \quad \Rightarrow \quad LB = BL_0\,.$$

Therefore, B maps eigenfunctions of L_0 to eigenfinctions of L_1.
The transformation from L_0 to L_1 is a one-step Darboux transformation. Iterating them, one constructs more complicated differentail operators from simpler ones, with explicit knowledge of eigenfunctions.
For our generalised KP hierarchy, one needs to replace ∂ by a generalised Bessel operator (Bakalov–Horozov–Yakimov):
$L_0 = p(D)\,, \quad D = x^{-m}\prod_{i=0}^{m-1}(x\partial - mk_i - i)\,.$
By factorizing L_0 into a product $L_0 = AB$ of two differential operators A, B with rational coefficients, we obtain $L_1 = BA$. In the process, L_1 will acquire additional (apparent) singularities.

More equations

Why not consider the full hierarchy?

$$\frac{\partial L}{\partial s_j} = [(L^j)_+, L] \quad (j = 1, 2, \dots)$$

(So previous time-variables t_j correspond to s_{mj}.)
Our previous solutions where of the form $L = MyM^{-1}$ with

$$M = 1 - \epsilon_0 w(X(\mathbf{t}) - x\mathtt{Id}_V)^{-1}(Y - y\mathtt{Id}_V)^{-1} v\epsilon_0 \,.$$

They had, in fact, the following symmetry property: $\epsilon_i M = M\epsilon_i$ and $\epsilon_i L = L\epsilon_{i+1}$.
The symmetry considerations show that such L cannot satisfy the extra equations of the hierarchy with $j \notin m\mathbb{N}$. Therefore, by turning on this extra time flows we should get more general solutions.

More solutions

It turns out that one needs to consider a bigger quiver Q_∞, obtained by adding not just one, but m arrows $v_i : \infty \to i$. Then the corresponding quiver data will be a collection of maps

$$X_i : V_i \to V_{i+1}\,, \quad Y_i : V_{i+1} \to V_i\,, \quad v_i : V_\infty \to V_i\,, \quad w_i : V_i \to V_\infty\,,$$

satisfying the relations of Π^λ:

$$X_{i-1}Y_{i-1} - Y_iX_i + v_iw_i = \lambda_i \mathtt{Id}_{V_i} \qquad (i \in \mathbb{Z}/m\mathbb{Z})$$

$$-\sum_{i=0}^{m-1} w_iv_i = \lambda_\infty \mathtt{Id}_{V_\infty}$$

As before, put $X = X_0 + \cdots + X_{m-1}$ and $Y = Y_0 + \cdots + Y_{m-1}$. The Calogero–Moser variety $C_{\lambda,\alpha}$ is defined similarly; it is smooth for generic λ_i and has a natural Poisson structure.

Proposition. The functions $h_j = \sum_{i=0}^{m-1} tr(w_{i+j}Y^jv_i)$, $j \geq 1$ are Poisson-commuting functions on $C_{\lambda,\alpha}$.

Define a map

$$\theta : C_{\lambda,\alpha} \longrightarrow \{\text{“pseudo-Dunkl” operators } L\}$$

by

$$(X, Y, \{v_i\}, \{w_i\}) \mapsto \ L = MyM^{-1}$$

with

$$M = 1 - \sum_{i,j=0}^{m-1} \epsilon_i w_i (X - x\mathtt{Id}_V)^{-1}(Y - y\mathtt{Id}_V)^{-1} v_j \epsilon_j \,.$$

Theorem. Under the map θ, the commuting Hamiltonain flows on $C_{\lambda,\alpha}$ defined by the functions $h_j = \sum_{i=0}^{m-1} tr(w_{i+j} Y^j v_i)$ are mapped into the time-flows of the generalised KP hierarchy $\frac{\partial L}{\partial s_j} = [(L^j)_+, L]$.

Remark: The integrable system defined by Hamiltonians h_j can be viewed as a spin version of the CM system for $W = S_n \wr \mathbb{Z}_m$.

Further directions

- Trigonometric and elliptic versions
- Multiplicative quiver varieties: quiver version of 2D Toda and Ruijsenaars–Schneider model; cyclotomic DAHA
- Cyclotomic version of Gaudin model and opers
- Other quivers