Recent progress in the algebro-geometric construction of non-abelian monopoles

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Algebraic Curves with Symmetries, their Jacobians & Integrable Dynamical Systems

In collaboration with V.Z. Enolski.

- Reduction of $F = *F$ (or static $V(\Phi) = 0$ with PS BC's)

$$L = -\frac{1}{2}\mathrm{Tr}\, F_{ij}F^{ij} + \mathrm{Tr}\, D_i\Phi\, D^i\Phi + V(\Phi)$$

- $*_{\mathbb{R}^3} F = \pm D\Phi$, $\qquad B_i = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk} F^{jk} = \pm D_i\Phi$

- A *monopole* of charge n

$$\left.\sqrt{-\frac{1}{2}\mathrm{Tr}\,\Phi(r)^2}\right|_{r\to\infty} \sim 1 - \frac{n}{2r} + O(r^{-2}), \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

- Monopoles $\leftrightarrow$ Nahm Data $\leftrightarrow$ Hitchin Data

Three $n \times n$ matrices $T_i(s)$ with $s \in [0, 2]$ satisfying the following:

N1 Nahm's equations
$$\frac{dT_i}{ds} = \frac{1}{2} \sum_{j,k=1}^{3} \epsilon_{ijk} [T_j, T_k].$$

N2 $T_i(s)$ is regular for $s \in (0, 2)$ and has simple poles at $s = 0, 2$. Residues form $su(2)$ irreducible n-dimensional representation.

N3 $T_i(s) = -T_i^\dagger(s), \qquad T_i(s) = T_i^t(2 - s).$

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$$L(\zeta) = T_1 + iT_2 - 2iT_3\zeta + (T_1 - iT_2)\zeta^2$$
$$M(\zeta) = -iT_3 + (T_1 - iT_2)\zeta$$

Nahm's eqn.
$$\frac{dT_i}{ds} = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j, T_k] \iff [\frac{d}{ds} + M, L] = 0.$$

- Solve Weyl equation (charge n $SU(2)$ monopoles) $\boldsymbol{V}_{2n\times 2n}$

$$\Delta^\dagger \boldsymbol{V} = i\left(1_{2n}\frac{d}{ds} + i\sum_{j=1}^{3} T_j(s)\otimes\sigma_j - \sum_{j=1}^{3} x_j 1_n\otimes\sigma_j\right)\boldsymbol{V}(\boldsymbol{x},s) = 0$$

- Reconstruction $\boldsymbol{V}\boldsymbol{\mu} = (\boldsymbol{v}_1, \boldsymbol{v}_2)$, $\int_0^2 \boldsymbol{v}_a^\dagger(\boldsymbol{x},s)\boldsymbol{v}_b(\boldsymbol{x},s)ds = \delta_{ab}$

$$\Phi(\boldsymbol{x})_{ab} = i\int_0^2 s\boldsymbol{v}_a^\dagger(\boldsymbol{x},s)\boldsymbol{v}_b(\boldsymbol{x},s)ds, \qquad a,b = 1,2$$

$$A_j(\boldsymbol{x})_{ab} = i\int_0^2 \boldsymbol{v}_a^\dagger(\boldsymbol{x},s)\frac{\partial}{\partial x_j}\boldsymbol{v}_b(\boldsymbol{x},s)ds, \quad i = 1,2,3$$

BPS Monopoles

$\Delta^\dagger \boldsymbol{v} = 0$

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$$\Delta \boldsymbol{w} = i\left(1_{2n}\frac{d}{ds} - i\sum_{j=1}^{3} T_j(s)\otimes\sigma_j + \sum_{j=1}^{3} x_j 1_n \otimes \sigma_j\right)\boldsymbol{w}(\boldsymbol{x}, s)$$

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$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}, \qquad \boldsymbol{n}\otimes\boldsymbol{2} = (\boldsymbol{n}-\boldsymbol{1})\oplus(\boldsymbol{n}+\boldsymbol{1})$

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Both ends: only 2 normalisable solutions

Integrals computed in closed form

$$\int \boldsymbol{v}_a^\dagger(\boldsymbol{x},s)\boldsymbol{v}_b(\boldsymbol{x},s)\mathrm{d}s = \boldsymbol{v}_a^\dagger(\boldsymbol{x},s)\mathcal{Q}^{-1}(\boldsymbol{x},s)\boldsymbol{v}_b(\boldsymbol{x},s)$$

$$\mathcal{Q}(\boldsymbol{x},z) = \frac{1}{r^2}\mathcal{H}(\boldsymbol{x})\mathcal{T}(z)\mathcal{H}(\boldsymbol{x}) - \mathcal{T}(z)$$

$$\mathcal{H}(\boldsymbol{x}) = \sum_{i=1}^{3} 1_n \otimes x_i\sigma_i, \qquad \mathcal{T}(z) = \imath \sum_{i=1}^{3} T_i(z)\otimes\sigma_i$$

$$\int s\boldsymbol{v}_a^\dagger \boldsymbol{v}_b \mathrm{d}s = \boldsymbol{v}_a^\dagger\, \mathcal{Q}^{-1}\left[s + 2\mathcal{H}(\boldsymbol{x})\frac{\mathrm{d}}{\mathrm{d}(r^2)}\right]\boldsymbol{v}_b$$

$$\int \boldsymbol{v}_a^\dagger \frac{\partial}{\partial x_i}\boldsymbol{v}_b \mathrm{d}s = \boldsymbol{v}_a^\dagger\ \mathcal{Q}^{-1}\left[\frac{\partial}{\partial x_i} + \mathcal{H}(\boldsymbol{x})\frac{s\,x_i + \imath(\boldsymbol{x}\times\nabla)_i}{r^2}\right]\boldsymbol{v}_b$$

Observe Dependence only on boundary values and their derivatives

A lesser known Ansatz of Nahm

$\Delta w = 0$

- $\boldsymbol{w} = (1_2 + \widehat{\boldsymbol{u}}(\boldsymbol{x}) \cdot \boldsymbol{\sigma})|\chi > \otimes\, \widehat{\boldsymbol{w}}(s)$ $\qquad |\chi >\notin \ker(1_2 + \widehat{\boldsymbol{u}}(\boldsymbol{x}) \cdot \boldsymbol{\sigma})$

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$$\Delta \boldsymbol{w} = 0 \iff \begin{cases} 0 = \left(\imath \dfrac{\mathrm{d}}{\mathrm{d}s} + \widehat{\boldsymbol{u}} \cdot \boldsymbol{R} \right) \widehat{\boldsymbol{w}}(s) \\ 0 = \mathcal{L}_k \widehat{\boldsymbol{w}}(s) := \left(\imath\, \widehat{u}^k \dfrac{\mathrm{d}}{\mathrm{d}s} + R^k + \imath(\boldsymbol{R} \times \widehat{\boldsymbol{u}})^k \right) \widehat{\boldsymbol{w}}(s) \end{cases}$$

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- $[\mathcal{L}_1, \mathcal{L}_2] = 0 \Longleftrightarrow$ *Nahm's Eqns* $\quad \& \quad \widehat{\boldsymbol{u}} \cdot \widehat{\boldsymbol{u}} = 1$

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- $[\mathcal{L}_1, \mathcal{L}_2] = 0 \Longleftrightarrow$ *Nahm's Eqns* $\quad \& \quad \widehat{\boldsymbol{u}} \cdot \widehat{\boldsymbol{u}} = 1$
- $\boldsymbol{y} = \left(\frac{1+\zeta^2}{2\,\imath}, \frac{1-\zeta^2}{2}, -\zeta \right) \qquad \boldsymbol{y} \cdot \overline{\boldsymbol{y}} = \frac{(1+|\zeta|^2)^2}{2} \qquad \boldsymbol{y} \cdot \boldsymbol{y} = 0$

$\widehat{\boldsymbol{u}} = \widehat{\boldsymbol{u}}(\zeta) := \imath \frac{\boldsymbol{y} \times \overline{\boldsymbol{y}}}{\boldsymbol{y} \cdot \overline{\boldsymbol{y}}}, \quad \mathrm{Re}(\boldsymbol{y}), \mathrm{Im}(\boldsymbol{y})$ orthogonal basis of $\mathbb{R}^3$

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$\Delta w = 0$

- $\widehat{\boldsymbol{u}} \cdot \boldsymbol{R} = M + \imath\,[(x_1 - \imath\, x_2)\zeta - \imath\, x_3] \qquad M := (T_1 - \imath\, T_2)\zeta - \imath\, T_3$
  $2\,\imath \boldsymbol{y} \cdot \boldsymbol{R} = L(\zeta) - \eta \qquad \eta := 2\boldsymbol{y} \cdot \boldsymbol{x}$

$$L(\zeta) := 2\,\imath\, \boldsymbol{y} \cdot \boldsymbol{T} = (T_1 + \imath\, T_2) - 2\,\imath\, T_3\,\zeta + (T_1 - \imath\, T_2)\,\zeta^2$$

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- $\Delta \boldsymbol{w} = 0 \Longleftrightarrow \begin{cases} 0 = (L(\zeta) - \eta)\, \widehat{\boldsymbol{w}}(s) \\ 0 = \left(\dfrac{\mathrm{d}}{\mathrm{d}s} + M\right) \widehat{\boldsymbol{w}}(s) \end{cases}$
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- $[\dfrac{d}{ds} + M(\zeta), L(\zeta)] = 0, \quad \mathcal{C}: \; 0 = \det(\eta 1_n - L(\zeta)) := P(\eta, \zeta)$

  $$P(\eta, \zeta) = \eta^n + a_1(\zeta)\eta^{n-1} + \ldots + a_n(\zeta), \qquad \deg a_r(\zeta) \leq 2r$$

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  Minitwistor description

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  - $\mathcal{C}_{\sigma-\text{model}} \subset \mathbb{P}^2 := \mathcal{S}$
  - $\mathcal{S} = T^*\Sigma$ Hitchin Systems on a Riemann surface Σ
  - $\mathcal{S} = K3$
  - $\mathcal{S}$ a Poisson surface
  - separation of variables $\leftrightarrow \mathrm{Hilb}^{[N]}(\mathcal{S})$
  - X the total space of an appropriate line bundle $\mathcal{L}$ over $\mathcal{S} \leftrightarrow$ noncompact CY

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- $\mathcal{C}$ has real structure: antiholomorphic involution which reverses orientation of lines $(\eta, \zeta) \to (-\bar{\eta}/\bar{\zeta}^2, -1/\bar{\zeta})$

  $a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-\frac{1}{\bar{\zeta}})} \Longrightarrow a_r(\zeta)$ given by $2r+1$ (real) parameters

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- genus given by Riemann Hurwitz formula $g_{\text{monopole}} = (n-1)^2$

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  $a_r(\zeta) = \chi_r \left[\prod_{l=1}^{r} \left(\frac{\overline{\alpha}_l}{\alpha_l} \right)^{1/2} \right] \prod_{k=1}^{r} (\zeta - \alpha_r)(\zeta + \frac{1}{\overline{\alpha}_r})$

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  $\alpha_r \in \mathbb{C}, \ \chi \in \mathbb{R}$ $a_r(\zeta)$ given by $2r+1$ (real) parameters
- reality constrains the form of the period matrix
- there may be between 0 and $g+1$ ovals of fixed points of the antiholomorphic involution.
- Imposing reality can be one of the hardest steps.

Let $\mathcal{C}$ be a smooth algebraic curve of genus $g_{\mathcal{C}}$ with $n \geqslant 1$ punctures P_j, $j = 1, \ldots, n$. Then for each set of $g_{\mathcal{C}} + n - 1$ points $\delta_1, \ldots, \delta_{g_{\mathcal{C}}+n-1}$ in general position, there exists a unique function $\Psi_j\,(t, P)$ and local coordinates $w_j(P)$ for which $w_j(P_j) = 0$, such that

1. The function Ψ_j of $P \in \mathcal{C}$ is meromorphic outside the punctures and has at most simple poles at δ_r (if all of them are distinct);
2. In the neighbourhood of the puncture P_l the function Ψ_j has the form (for $i \in \mathbb{N}^+$, $w_l = w_l\,(P)$)

$$\Psi_j\,(s, P) = e^{s\, w_l{}^{-i}} \left(\delta_{jl} + \sum_{k=1}^{\infty} \alpha_{jl}^k\,(s)\, w_l^k\right)$$

Meromorphic differential describe flows

$$w_j(P_j) = 0 \qquad d\Omega^{(i)} = d\left(w_j^{-i} + 0\,(w_j)\right) \qquad \oint_{\mathfrak{a}_k} d\Omega^{(i)} = 0$$

Monopole Reconstruction

- Solutions of $\Delta w = 0$ in terms of joint eigenfunctions

$$(\eta 1_n - L(\zeta))\,\hat{w} = 0$$
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- Solve $\hat{w}$ in terms of Baker-Akhiezer function.
- $M(\zeta) = -iT_3 + (T_1 - iT_2)\zeta$ poles at $\zeta = \infty$

$$\frac{P(\eta,\zeta)}{\zeta^{2n}} \sim \prod_{j=1}^{n} \left(\frac{\eta}{\zeta^2} - \rho_j\right) \qquad \frac{\eta}{\zeta} = \rho_j \zeta,\ \zeta \sim \infty_j \text{ } n \text{ points on } \mathcal{C}$$

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Monopole Reconstruction

- Solutions of $\Delta w = 0$ in terms of joint eigenfunctions

$$(\eta 1_n - L(\zeta))\,\hat{w} = 0$$
$$\left(\frac{d}{ds} + M(\zeta)\right)\hat{w} = 0$$

- Solve $\hat{w}$ in terms of Baker-Akhiezer function.
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- $\exists!$ meromorphic differential $\gamma_\infty \equiv \gamma_\infty(P) = \left(\frac{\rho_j}{t^2} + O(1)\right)\mathrm{d}t$,

as $P \to \infty_j$, $\oint_{\mathfrak{a}_k} \gamma_\infty(P) = 0$, $\quad \boldsymbol{U} = \frac{1}{2i\pi}\oint_{\mathfrak{b}} \gamma_\infty$

H1 $\mathcal{C} \subset T\mathbb{P}^1$ Reality conditions $a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-\frac{1}{\bar{\zeta}})}$

H2 $\mathcal{L}^\lambda(m)$ the holomorphic line bundle on $T\mathbb{P}^1$ with transition function $g_{01} = \zeta^m \exp\left(-\lambda\eta/\zeta\right)$.
$\mathcal{L}^\lambda := \mathcal{L}^\lambda(0), \qquad \mathcal{L}^\lambda(m) \equiv \mathcal{L}^\lambda \otimes \pi^*\mathcal{O}(m)$
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$\mathcal{O}(\mathcal{L}^s) \hookrightarrow \mathcal{O}(\mathcal{L}^s(n-2)) \times$ a section of $\pi^*\mathcal{O}(n-2)|_{\mathcal{C}}$

The Ercolani-Sinha Constraints

- $\mathcal{L}^2$ trivial $\Longrightarrow f_0(\eta,\zeta) = \exp\left\{-2\frac{\eta}{\zeta}\right\} f_1(\eta,\zeta)$

$$\operatorname{dlog} f_0 = \mathrm{d}\left(-2\frac{\eta}{\zeta}\right) + \operatorname{dlog} f_1, \quad \exp\oint_\gamma \operatorname{dlog} f_0 = 1 \quad \forall \gamma \in H_1(\mathbb{Z}, \mathcal{C})$$

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  3. $\exists$ 1-cycle $\mathfrak{es} = \boldsymbol{n}\cdot\mathfrak{a} + \boldsymbol{m}\cdot\mathfrak{b}$ s.t. for every holomorphic differential
  $$\Omega = \frac{\beta_0\eta^{n-2} + \beta_1(\zeta)\eta^{n-3} + \ldots + \beta_{n-2}(\zeta)}{\frac{\partial\mathcal{P}}{\partial\eta}}\, d\zeta, \quad \oint_{\mathfrak{es}} \Omega = -2\beta_0$$

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- **H3** $\mathcal{L}^s$ trivial $\Longleftrightarrow s\,\boldsymbol{U}\in\Lambda, \qquad 2\boldsymbol{U}$ is a primitive vector in Λ

- Baker-Akhiezer function

$$\Psi_j(s,P) = g_j(P)\,\frac{\theta(\phi(P) + s\boldsymbol{U} - \mu_j)}{\theta(\phi(P) - \mu_j)}\, e^{s\int_{P_0}^{P}\gamma_\infty} \times \frac{\theta(\phi(P_j) - \mu_j)}{\theta(\phi(P_j) + s\boldsymbol{U} - \mu_j)}$$

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Monopole Reconstruction

Reconstruction $\Delta^{\dagger}\boldsymbol{V} = 0 \qquad \boldsymbol{V}\mu = (\boldsymbol{v}_1, \boldsymbol{v}_2)$

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Problem: extract norm. solns.

Example: $n = 1$

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Example: $n = 2$

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- $\mathcal{C} : \{(\zeta,\eta) \mid \eta^2 + \frac{K^2}{4}(\zeta^4 + 2(1-2k'^2)\zeta^2 + 1) = 0\} \quad \tau = \imath\frac{K(k')}{K(k)}$
- $\mathfrak{I} : \ \zeta_k(\boldsymbol{x}) \to -1/\overline{\zeta}_k(\boldsymbol{x}), \quad \eta_k(\boldsymbol{x}) \to -\overline{\eta}_k(\boldsymbol{x})/\overline{\zeta}_k^2(\boldsymbol{x})$
  $$(\zeta_1(\boldsymbol{x}), \zeta_2(\boldsymbol{x}), \zeta_3(\boldsymbol{x}), \zeta_4(\boldsymbol{x})) = \left(\zeta_1(\boldsymbol{x}),\ \zeta_2(\boldsymbol{x}),\ -\frac{1}{\overline{\zeta_1(\boldsymbol{x})}},\ -\frac{1}{\overline{\zeta_2(\boldsymbol{x})}}\right)$$

Example: $n = 2$

- $T_j(s) = -\frac{\imath}{2} f_j(s)\sigma_j, \quad f_1(s) = K\frac{\mathrm{dn}(Ks;k)}{\mathrm{cn}(Ks;k)} = \frac{\pi\vartheta_2(0)\vartheta_3(0)}{2}\frac{\vartheta_3(s/2)}{\vartheta_2(s/2)}$

  Nahm's eqns. $\equiv$ Spinning Top
- $\mathcal{C} : \{(\zeta,\eta) \mid \eta^2 + \frac{K^2}{4}(\zeta^4 + 2(1-2k'^2)\zeta^2 + 1) = 0\} \ \tau = \imath\frac{K(k')}{K(k)}$
- $\mathfrak{I} : \ \zeta_k(\boldsymbol{x}) \to -1/\overline{\zeta}_k(\boldsymbol{x}), \quad \eta_k(\boldsymbol{x}) \to -\overline{\eta}_k(\boldsymbol{x})/\overline{\zeta}_k^2(\boldsymbol{x})$

$$(\zeta_1(\boldsymbol{x}), \zeta_2(\boldsymbol{x}), \zeta_3(\boldsymbol{x}), \zeta_4(\boldsymbol{x})) = \left(\zeta_1(\boldsymbol{x}),\ \zeta_2(\boldsymbol{x}),\ -\frac{1}{\overline{\zeta_1(\boldsymbol{x})}},\ -\frac{1}{\overline{\zeta_2(\boldsymbol{x})}}\right)$$

- $\overline{\int_{\infty_1}^{P_1} v} = -\int_{\infty_1}^{P_3} v - \frac{\tau}{2}$

$$\overline{\vartheta_1\left(\int_{\infty_1}^{P_{1,2}} v\right)} = -\imath\vartheta_4\left(\int_{\infty_1}^{P_{3,4}} v\right)\exp\left\{-\imath\pi\int_{\infty_1}^{P_{3,4}} v - \frac{\imath\pi\tau}{4}\right\}$$

Example: $n = 2$

The Weyl Equation

$$0 = \left(\frac{\mathrm{d}}{\mathrm{d}s} + M(s,\zeta)\right)\widehat{\boldsymbol{w}}(s,\zeta)$$

$$= \left[1_2\frac{\mathrm{d}}{\mathrm{d}s} + \frac{1}{2}\begin{pmatrix} -f_3(s) & -\imath\zeta(f_1(s) - f_2(s)) \\ -\imath\zeta(f_1(s) + f_2(s)) & f_3(s) \end{pmatrix}\right]\begin{pmatrix}\widehat{w}_1(s,\zeta) \\ \widehat{w}_2(s,\zeta)\end{pmatrix}$$

$$0 = \left(\frac{\mathrm{d}}{\mathrm{d}s} + M(s,\zeta)\right)\widehat{\boldsymbol{w}}(s,\zeta)$$
$$= \left[1_2\frac{\mathrm{d}}{\mathrm{d}s} + \frac{1}{2}\begin{pmatrix} -f_3(s) & -\imath\zeta(f_1(s)-f_2(s)) \\ -\imath\zeta(f_1(s)+f_2(s)) & f_3(s)\end{pmatrix}\right]\begin{pmatrix}\widehat{w}_1(s,\zeta) \\ \widehat{w}_2(s,\zeta)\end{pmatrix}$$

$$C^{-1}(s)\left[T_1(s) - \imath T_2(s)\right]C(s) = \frac{k'}{2k}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$ using Nahm's eqns.

$$\left[1_2\frac{\mathrm{d}}{\mathrm{d}s} + \begin{pmatrix}0 & f_3(s) \\ f_3(s) & 0\end{pmatrix}\right]\Phi = -\zeta\,\frac{k'}{2k}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\Phi, \quad \Phi := C^{-1}\widehat{\boldsymbol{w}}$$

$$0 = \left(\frac{\mathrm{d}}{\mathrm{d}s} + M(s,\zeta)\right)\widehat{\boldsymbol{w}}(s,\zeta)$$

$$= \left[1_2\frac{\mathrm{d}}{\mathrm{d}s} + \frac{1}{2}\begin{pmatrix} -f_3(s) & -\imath\zeta(f_1(s)-f_2(s)) \\ -\imath\zeta(f_1(s)+f_2(s)) & f_3(s)\end{pmatrix}\right]\begin{pmatrix}\widehat{w}_1(s,\zeta)\\ \widehat{w}_2(s,\zeta)\end{pmatrix}$$

$$C^{-1}(s)\left[T_1(s) - \imath T_2(s)\right]C(s) = \frac{k'}{2k}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$ using Nahm's eqns.

$$\left[1_2\frac{\mathrm{d}}{\mathrm{d}s} + \begin{pmatrix}0 & f_3(s)\\ f_3(s) & 0\end{pmatrix}\right]\Phi = -\zeta\,\frac{k'}{2k}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Phi, \quad \Phi := C^{-1}\widehat{\boldsymbol{w}}$$

$$\frac{\mathrm{d}^2\Psi(u,v)}{\mathrm{d}u^2} - \mathfrak{P}(u)\Psi(u,v) = \lambda\Psi(u,v) \qquad \mathfrak{P}(u) := 2\wp\left(u \,\middle|\, \frac{\omega}{2},\omega'\right)$$

Curves related by Landen Transformation

Example: $n = 2$

$|\chi\rangle = |1, 0\rangle^T \qquad \hat{\boldsymbol{w}}^{(m)}(\boldsymbol{x}, s) = (1_2 \otimes C(s)) \cdot \phi_m \cdot \mathcal{D}_m$

Example: $n = 2$

$$|\chi\rangle = |1,0\rangle^T \qquad \hat{\boldsymbol{w}}^{(m)}(\boldsymbol{x}, s) = (1_2 \otimes C(s)) \cdot \phi_m \cdot \mathcal{D}_m$$

$$\phi_m = \begin{pmatrix} 1 \\ \imath\zeta_m \end{pmatrix} \otimes \begin{pmatrix} -\vartheta_3(\alpha_m)\dfrac{\vartheta_2(\alpha_m - s/2)}{\vartheta_2(s/2)} \\ \\ \vartheta_1(\alpha_m)\dfrac{\vartheta_4(\alpha_m - s/2)}{\vartheta_2(s/2)} \end{pmatrix} \qquad \alpha_m = \int_{\infty_1}^{P_m} v$$

Example: $n = 2$

$$|\chi\rangle = |1,0\rangle^T \qquad \hat{\boldsymbol{w}}^{(m)}(\boldsymbol{x}, s) = (1_2 \otimes C(s)) \cdot \phi_m \cdot \mathcal{D}_m$$

$$\phi_m = \begin{pmatrix} 1 \\ \imath\zeta_m \end{pmatrix} \otimes \begin{pmatrix} -\vartheta_3(\alpha_m)\dfrac{\vartheta_2(\alpha_m - s/2)}{\vartheta_2(s/2)} \\ \\ \vartheta_1(\alpha_m)\dfrac{\vartheta_4(\alpha_m - s/2)}{\vartheta_2(s/2)} \end{pmatrix} \qquad \alpha_m = \int_{\infty_1}^{P_m} v$$

$$\mathcal{D}_m = \frac{\vartheta_1\left(\frac{1}{4}\right)\vartheta_3\left(\frac{1}{4}\right)\exp\{s(\beta_m - \imath(x_1 - x_2)\zeta_m - \imath x_3 - x_4)\}}{\theta_1\left(\alpha_m - \frac{1}{4}\right)\theta_4\left(\alpha_m + \frac{1}{4}\right)\vartheta_3(0)(1 + |\zeta_m|^2)}$$

$$\beta_m = \frac{1}{4}\left\{\frac{\vartheta_1'(\alpha_m|\tau)}{\vartheta_1(\alpha_m|\tau)} + \frac{\vartheta_3'(\alpha_m|\tau)}{\vartheta_3(\alpha_m|\tau)} - \imath\pi\right\}$$

Example: $n = 2$

$$|\chi\rangle = |1,0\rangle^T \qquad \hat{\boldsymbol{w}}^{(m)}(\boldsymbol{x}, s) = (1_2 \otimes C(s)) \cdot \phi_m \cdot \mathcal{D}_m$$

$$\phi_m = \begin{pmatrix} 1 \\ \imath\zeta_m \end{pmatrix} \otimes \begin{pmatrix} -\vartheta_3(\alpha_m)\dfrac{\vartheta_2(\alpha_m - s/2)}{\vartheta_2(s/2)} \\ \\ \vartheta_1(\alpha_m)\dfrac{\vartheta_4(\alpha_m - s/2)}{\vartheta_2(s/2)} \end{pmatrix} \qquad \alpha_m = \int_{\infty_1}^{P_m} v$$

$$\mathcal{D}_m = \frac{\vartheta_1\left(\frac{1}{4}\right)\vartheta_3\left(\frac{1}{4}\right)\exp\{s(\beta_m - \imath(x_1 - x_2)\zeta_m - \imath x_3 - x_4)\}}{\theta_1\left(\alpha_m - \frac{1}{4}\right)\theta_4\left(\alpha_m + \frac{1}{4}\right)\vartheta_3(0)(1 + |\zeta_m|^2)}$$

$$\beta_m = \frac{1}{4}\left\{\frac{\vartheta_1'(\alpha_m|\tau)}{\vartheta_1(\alpha_m|\tau)} + \frac{\vartheta_3'(\alpha_m|\tau)}{\vartheta_3(\alpha_m|\tau)} - \imath\pi\right\}$$

$$C(s) = \begin{pmatrix} F(s) & G(s) \\ G(s) & F(s) \end{pmatrix}$$

$$G^2(s) = \tfrac{1}{2}\left(\frac{\mathrm{dn}(Ks;k)}{\mathrm{cn}(Ks;k)} - 1\right), \; 2F(s)G(s) = k'\frac{\mathrm{sn}(Ks;k)}{\mathrm{cn}(Ks;k)}$$

Example: $n = 2$

$$|\chi\rangle = |1,0\rangle^T \qquad \boldsymbol{w}^{(m)}(\boldsymbol{x},z) = (1_2 \otimes C(s)) \cdot \phi_m \cdot \mathcal{D}_m$$

$$\phi_m = \begin{pmatrix} 1 \\ \imath\zeta_m \end{pmatrix} \otimes \begin{pmatrix} -\vartheta_3(\alpha_m)\dfrac{\vartheta_2(\alpha_m - s/2)}{\vartheta_2(s/2)} \\ \\ \vartheta_1(\alpha_m)\dfrac{\vartheta_4(\alpha_m - s/2)}{\vartheta_2(s/2)} \end{pmatrix} \qquad \alpha_m = \int_{\infty_1}^{P_m} v$$

$$\det\phi = \frac{{\vartheta_1'}^2(0)\prod_{1\le i<j\le 4}\vartheta_1(\alpha_i - \alpha_j)}{\pi^2\prod_{k=1}^4 \vartheta_2(\alpha_k)\vartheta_4(\alpha_k)}$$

$$\boldsymbol{W}^\dagger \boldsymbol{W} = \mathcal{D}^\dagger.\mathcal{G}.\mathcal{D}, \qquad \mathcal{G} = (\mathcal{G}_{i,j})$$

$$\mathcal{G}_{i,j} = (1 + \bar{\zeta}_i\zeta_j)\,\vartheta_3(0)\,\vartheta_2\left(\frac{s}{2}\right)\,\vartheta_3\left(\alpha_j + \alpha_{\mathfrak{J}(i)}\right)\vartheta_2\left(\alpha_j - \alpha_{\mathfrak{J}(i)} - \frac{1}{2}s\right)$$