Dynamics of Dicke model via generalized theta-functions

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Dicke model: formulation

System of N two-level non-interacting atoms + single-mode electromagnetic field

- resonance approximation: $\Omega_{Atom} \approx \omega_F \Rightarrow$ two-level approximation
- dipole approximation: $\lambda \gg a \Rightarrow \mathcal{H}_{Int} = \vec{d}\vec{E}$
- rotating-wave approximation (RWA)

Hamiltonian

$$\mathcal{H} = \sum_{j=1}^{N} \Omega_j S_j^z + \omega b^+ b + g \sum_{j=1}^{N} \left(b^+ S_j^- + b S_j^+ \right)$$

- S_j^z – inverse level population, $b^+ b$ – "number of photons"
- $\vec{d}_j = \vec{d}_{21} S_j^+ + \vec{d}_{12} S_j^-$ – dipole momentum of j-th atom

Classical ("semi-classical") model:

- Physically: atom has discrete levels, field is classical
- Dynamics of average values is in focus

Dicke model: integrability

Dimension of phase space

- N atoms, each is described by set of 3 variables $\{S_j^x,\ S_j^y,\ S_j^z\}$ + constraint $(S_j^x)^2 + (S_j^y)^2 + (S_j^z)^2 = const \Rightarrow$
  $2N$ independent variables describe atomic subsystem
- b^+, b – 2 field variables
- $\Rightarrow$ dim of phase space $= 2N + 2$

Integrals of motion

- due to classical Liouville theorem $N + 1$ integrals of motion in involution are required for this system to be integrable
- we have only one "obvious" integral:

$$\mathcal{H}_{N+1} = \sum_{j=1}^{N} S_j^z + b^+ b$$

Dicke model: integrability

Lax formulation of Dicke model

$$L(\lambda) = \begin{bmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & -\alpha(\lambda) \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{2}{g}\lambda - \frac{\omega}{g^2} + \sum_j \frac{S_j^z}{\lambda-\varepsilon_j} & \frac{2}{g} b + \sum_j \frac{S_j^-}{\lambda-\varepsilon_j} \\ \frac{2}{g} b^+ + \sum_j \frac{S_j^+}{\lambda-\varepsilon_j} & -\left(\frac{2}{g}\lambda - \frac{\omega}{g^2} + \sum_j \frac{S_j^z}{\lambda-\varepsilon_j}\right) \end{bmatrix}$$

Invariant function $det\left[L(\lambda) - \mu\mathbf{1}\right] = 0$ is generating function for integrals of motion:

$$\begin{aligned} \mu^2(\lambda) &= \alpha^2(\lambda) + \beta(\lambda)\gamma(\lambda) = \\ &= \frac{1}{g^4}(2\lambda-\omega)^2 + \frac{4}{g}\mathcal{H}_{N+1} + \frac{2}{g^2}\sum_{k=1}^{N}\frac{\mathcal{H}_k}{(\lambda-\varepsilon_k)} + \sum_{k=1}^{N}\frac{s_k^2}{(\lambda-\varepsilon_k)^2}, \end{aligned}$$

$$\mathcal{H}_k = \Delta_k S_k^z + g\left(b^+S_k^- + bS_k^+\right) + \sum_{i\neq k}\frac{g^2}{\varepsilon_k-\varepsilon_i}\left[S_k^z S_i^z + \frac{1}{4}(S_k^+S_i^- + S_k^-S_i^+)\right]$$

$$\mathcal{H}_{N+1} = \sum_{j=1}^{N} S_j^z + b^+b, \quad \Delta_k = (2\varepsilon_k - \omega) \quad \Rightarrow \quad \boxed{\mathcal{H} = \omega\mathcal{H}_{N+1} + \sum \mathcal{H}_j}$$

Integrability due to Liouville theorem

- Dynamical system:

$$\frac{d}{dt}S_i^a = \{S_i^a, \mathcal{H}\}, \qquad \frac{d}{dt}b^{\pm} = \{b^{\pm}, \mathcal{H}\}$$

$$\{S_i^a, S_k^b\} = \delta_{ik}\varepsilon^{abc}S_i^c, \qquad \{b^-, b^+\} = \imath$$

$$S_i^2 = const, \ i = 1..N$$

- **Phase space** $\mathcal{M}^{2N+2} = \underbrace{S^2 \times S^2 \ldots \times S^2}_{N} \times \mathbb{R}^2$ is a product of N (Bloch) spheres and 2d plane; its dim $= 2N+2$
- there exist $N+1$ integrals of motion $\mathcal{H}_k$ in involution $\{\mathcal{H}_i, \mathcal{H}_k\} = 0$
- $\Rightarrow$ **due to Liouville theorem this system is integrable** and joint surface of level of integrals of motion $\{\mathcal{H}_i = h_i = const\} \simeq T^{N+1} = \underbrace{S^1 \times S^1 \ldots \times S^1}_{N+1}$ is Liouville torus

Algebraic integrability

- Real torus of algebraically integrable system allows analytic continuation to complex algebraic torus related to the algebraic curve.
- Here Lax matrix gives a curve $\mu^2(\lambda)$, corresponding Abelian variety is the Jacobian of this curve and dynamics linearizes on it.

Algebraic integrability

Standard case

$$T^N \to T^N_{\mathbb{C}}$$

$$T^N_{\mathbb{C}} = Symm(\underbrace{\mathcal{R} \times \mathcal{R} \ldots \times \mathcal{R}}_{N}) = Jac(\mathcal{R}),$$

$\mathcal{R}$ – Riemann surface which corresponds to the algebraic curve of genus N, $Jac(\mathcal{R}) = \mathbb{C}^g / (\mathbb{Z}^g + B\mathbb{Z}^g)$ – Jacobian of Riemann surface $\mathcal{R}$ with Riemann matrix B, $\mathrm{Re}B > 0$, constructed from b-periods of $\mathcal{R}$

- N variables λ_i are introduced on curve μ^2 (or one variable on each copy of $\mathcal{R}$) and N canonically conjugated to them $\mu_i = \mu(\lambda_i)$
- Jacobi inversion problem arises: to inverse the map

  $$A: \ S^g\mathcal{R} \to Jac(\mathcal{R}),$$

  or, equivalently, for some point $\vec{z} = (z_1, \ldots, z_g) \in Jac(\mathcal{R})$ find g points $P_1, \ldots, P_g$ of Riemann surface $\mathcal{R}$ such that:

  $$\sum_{k=1}^{N} \int_{P_0}^{P_k} \omega_j = z_j, \qquad j = 1 \ldots, g.$$

Dicke model: integrability

Standard approach

- For this particular system we have:

$$T^{N+1} \to T^N_{\mathbb{C}}$$

  as number of initial variables is $2(N+1)$ and genus of the Riemann surface corresponding to the algebraic curve is N
- So we need to fix one of the variables (and its conjugated) in order to be able to solve the Jacobi inversion problem for N variables in terms of standard theta-functions
- To recover the remaining variable we need to perform additional integration: e.g., in terms of hyperelliptic ζ- and σ-functions (e.g., J.C. Eilbeck, V.Z. Enolskii, H. Holden, 2002)

Dicke model: integrability

$$T^{N+1} \to \tilde{T}_{\mathbb{C}}^{N+1}$$

$$T_{\mathbb{C}}^{N} = Symm(\underbrace{\mathcal{R} \times \mathcal{R} \ldots \times \mathcal{R}}_{N+1}) = GJac(\mathcal{R}),$$

$\mathcal{R}$ – Riemann surface which corresponds to the algebraic curve

$$\mu^2(\lambda) = \frac{1}{g^4}(2\lambda - \omega)^2 + \frac{4}{g}\mathcal{H}_{N+1} + \frac{2}{g^2}\sum_{k=1}^{N}\frac{\mathcal{H}_k}{(\lambda - \varepsilon_k)} + \sum_{k=1}^{N}\frac{s_k^2}{(\lambda - \varepsilon_k)^2} \quad \to$$

$$w^2(\lambda) = Q_{2N+2}(\lambda),$$

$GJac(\mathcal{R})$ – generalized Jacobian of $\mathcal{R}$.

- w^2 is an even (hyperelliptic) algebraic curve of genus N

Strategy

- Construct $N+1$ separated variables (and $N+1$ conjugated to them)
- Use extended Jacobi-Abel map
- Use theory of generalized theta-functions in order to solve inversion problem

Separation of variables

$$L(\lambda) = \begin{bmatrix} \frac{2}{g}\lambda - \frac{\omega}{g^2} + \sum_j \frac{S_j^z}{\lambda - \varepsilon_j} & \frac{2}{g} b + \sum_j \frac{S_j^-}{\lambda - \varepsilon_i} \\ \frac{2}{g} b^+ + \sum_j \frac{S_j^+}{\lambda - \varepsilon_i} & -\alpha(\lambda) \end{bmatrix} \to \widetilde{L}(\lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & -a(\lambda) \end{bmatrix}$$

$$a(\lambda) = \frac{\beta+\gamma}{2}, \quad b(\lambda) = \frac{\beta-\gamma}{2} - \alpha, \quad c(\lambda) = \frac{\gamma-\beta}{2} - \alpha$$

Generalized coordinates λ_k – zeroes of the rational function $c(\lambda)$

$$c(\lambda) = 0 \ \Leftrightarrow\ \frac{b^+ - b}{g} + \frac{\omega}{g^2} \sum_j + \frac{\imath S_j^y - S_j^z}{\lambda_k - \varepsilon_j} - \frac{2}{g}\lambda_k = 0$$

linear system of equations connects λ_k with initial variables

"Generalized momenta" μ_k are defined from the equation of the algebraic curve

$$\mu_j \equiv \mu(\lambda_j) = \frac{1}{g^4}(2\lambda_j - \omega)^2 + \frac{4}{g}\mathcal{H}_{N+1} + \frac{2}{g^2}\sum_{k=1}^{N} \frac{\mathcal{H}_k}{(\lambda_j - \varepsilon_k)} + \sum_{k=1}^{N} \frac{s_k^2}{(\lambda_j - \varepsilon_k)^2}$$

Dynamical variables in terms of separated variables

$$(\imath S_i^y - S_i^z) = -\frac{2}{g}\frac{\prod_{k=1}^{N+1}(\varepsilon_i - \lambda_k)}{\prod_{k=1,k\neq i}^{N}(\varepsilon_i - \varepsilon_k)} \equiv X_i, \quad S_i^z = \frac{1}{2}\left[X_i^2 + \frac{1}{X_i}\left(s_i^2 - (S_i^x)^2\right)\right]$$

$$\hat{S}_i^x = -\frac{\prod_{k=1}^{N+1}(\varepsilon_i - \lambda_k)}{\prod_{k=1,k\neq i}^{N}(\varepsilon_i - \varepsilon_k)}\sum_{k=1}^{N+1}\frac{\prod_{l=1,l\neq j}^{N}(\lambda_k - \varepsilon_l)}{\prod_{i=1,i\neq k}^{N+1}(\lambda_k - \lambda_i)}\mu_k, \quad \mu_k = \frac{\sqrt{\prod(\lambda - E_i)}}{\prod(\lambda - \varepsilon_j)}$$

$$b^+ - b = 2g\left[\sum_{j=1}^{N+1}\lambda_j - \sum_{j=1}^{N}\varepsilon_j\right] - \frac{\omega}{g}, \quad b^+ + b = g\sum_{k=1}^{N+1}\frac{\prod_{l=1}^{N}(\lambda_k - \varepsilon_l)}{\prod_{i=1,i\neq k}^{N+1}(\lambda_k - \lambda_i)}\mu_k$$

Equations for separated variables

Evolution of λ_k:

$$\frac{\partial \lambda_k}{\partial t} = \{\lambda_k, \mathcal{H}\} = \imath \frac{\partial \mathcal{H}}{\partial \mu_k} \Rightarrow$$

$$\frac{\partial \lambda_k}{\partial \tau} = \frac{\sum_{j=1, j\neq k}^{N+1} \lambda_j - \tilde{c}(N)}{\prod_{j=1}^{N+1} (\lambda_k - \lambda_j)} \sqrt{Q_{2N+2}(\lambda_k)},$$

where $\tilde{c}(N) = \sum_{j=1}^{N} \varepsilon_j + \frac{g^2 \omega}{4}$, $\tau = -\frac{4}{g} \imath t$

$$\sum_{k=1}^{N} \frac{\lambda_k^{i-1} \mathrm{d}\lambda_k}{\sqrt{Q_{2N}(\lambda_k)}} = dx_i, \qquad i = 1 \ldots, N+1,$$

$$\vec{x}^{\top} = (c_1, \ldots, c_{N-1}, \tau + c_N, \tau \tilde{c}(N) + c_{N+1})$$

Extended Abel-Jacobi map

$$\sum_{k=1}^{N} \int_{P_0}^{\lambda_k} \frac{\lambda^{i-1} \mathrm{d}\lambda}{\sqrt{Q_{2N}(\lambda)}} = x_i, \qquad i = 1 \ldots, N+1$$

- On Riemann surface of genus g there exist g independent holomorphic differentials (Abelian differentials of the first kind)
- Thus first $N-1$ integrals are holomorphic, while the last is the integral of the third kind with poles $\pm\infty$ and residues ± 1.
- Therefore, this system can be written in the canonical form:

$$\sum_{j=1}^{N} \int_{P_0}^{P_j} \boldsymbol{\omega}(P) = \boldsymbol{z}, \qquad P = (\lambda, \mu)$$

$$\sum_{j=1}^{N} \int_{P_0}^{P_j} \Omega_{\infty,\infty_-}(P) = Z,$$

- Extended Abel-Jacobi map: $S^{g+1}(\mathcal{R}) \rightarrow GJac(\mathcal{R})$

θ-functions with half-integer characteristics

- $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g)^T$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_g)^T$, $\alpha_i, \beta_j = 1/2$ or 0,

$$\theta \left[\begin{array}{c} \boldsymbol{\alpha}^T \\ \boldsymbol{\beta} \end{array} \right] (\boldsymbol{z}; \tau) = \sum_{\boldsymbol{m} \in \mathbb{Z}^g} \exp \left\{ (\boldsymbol{m} + \boldsymbol{\alpha})^T \tau (\boldsymbol{m} + \boldsymbol{\alpha}) + 2\imath\pi (\boldsymbol{m} + \boldsymbol{\alpha})(\boldsymbol{z} + \boldsymbol{\beta}) \right\}$$

Generalized θ-functions for our case

$$\tilde{\theta}(\boldsymbol{z}, Z) = \theta[\boldsymbol{K}](\boldsymbol{z} + \boldsymbol{q})\mathrm{e}^{Z/2} - \theta[\boldsymbol{K}](\boldsymbol{z} - \boldsymbol{q})\mathrm{e}^{-Z/2}$$

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- "Root" functions of the 1-st type (due to Yu. Fedorov, 1999):

$$\sqrt{(E_k - \lambda_1) \cdots (E_k - \lambda_N)} =$$
$$= C_k \frac{\mathrm{e}^{-Z/2} \theta[\boldsymbol{K} + \mathfrak{A}_k](\boldsymbol{z} + \boldsymbol{q}) - \mathrm{e}^{Z/2} \theta[\boldsymbol{K} + \mathfrak{A}_k](\boldsymbol{z} - \boldsymbol{q})}{\theta[\boldsymbol{K}](\boldsymbol{z})} \equiv \mathcal{F}_k,$$

where $\mathfrak{A}_k = \int_{P_0}^{E_k} \boldsymbol{\omega}$; C_k are some definite constants.

- Elementary symmetric polynomials on $\{\lambda_k\}$ can be expressed in terms of such a function:

$$s_k^{[N+1]} = u_k^{[N+1]} - \sum_{i=1}^{N+1} V_i^{[N+1]} \tilde{u}_{i-1}^{[N+1]} \mathcal{F}_i,$$

де $s_k^{[N+1]}$ та $u_k^{[N+1]}$ – elementary symmetric polynomials of k-th degree depending on $N+1$ variables $\{\lambda_k\}$ and $\{E_k\}$, respectively, in polynomial $\tilde{w}_{i-1}^{[N+1]}$ $(i-1)$-th variable is omitted, $V_i^{N+1} = \frac{1}{\prod_{j\neq i}(E_i - E_j)}$.

- Root functions of the 2-nd kind allow to find initial dynamical variables which contain μ_k; e.g.,

$$C_j \equiv \sum_{k=1}^{N+1} \frac{\mu_k}{(\lambda_k - E_j)\prod_{i=1}^{N+1}(\lambda_k - \lambda_i)} =$$

$$= \sum_{i=N+1}^{2N+1} \left(\prod_{j\neq i}(E_i - E_j)\right)^{-1} \left[E_{2N+2} - E_i + \sum_{k=1}^{N+1} \mathcal{F}_k / (\prod_{j\neq i}(E_i - E_j))\right] \frac{\Phi_{ij}\mathcal{F}_j}{\mathcal{F}_i \mathcal{F}_{2N+2}}$$

$$S_j^z = -\frac{1}{2}\left[\frac{s_j^2}{X_j} + X_j\{1 - \frac{g2}{4} C_j^2\}\right]$$

Genus 1 (expressions needed for $\imath S_i^y - S_i^z$ and $b^+ - b$)

$$\sqrt{(E_1-\lambda_1)(E_1-\lambda_2)} = \varkappa_1 \frac{\mathrm{e}^{-Z/2}\vartheta_4(q+z) - \mathrm{e}^{Z/2}\vartheta_4(q-z)}{\vartheta_1(z)},$$

$$\sqrt{(E_2-\lambda_1)(E_2-\lambda_2)} = \varkappa_2 \frac{\mathrm{e}^{-Z/2}\vartheta_3(q+z) - \mathrm{e}^{Z/2}\vartheta_3(q-z)}{\vartheta_1(z)},$$

$$\sqrt{(E_3-\lambda_1)(E_3-\lambda_2)} = \varkappa_3 \frac{\mathrm{e}^{-Z/2}\vartheta_2(q+z) - \mathrm{e}^{Z/2}\vartheta_2(q-z)}{\vartheta_1(z)},$$

$$\sqrt{(E_4-\lambda_1)(E_4-\lambda_2)} = \varkappa_4 \frac{\mathrm{e}^{-Z/2}\vartheta_1(q+z) - \mathrm{e}^{Z/2}\vartheta_1(q-z)}{\vartheta_1(z)},$$

here

$$\varkappa_1 = \frac{\vartheta_4(0)}{\vartheta_1(q)} \frac{\sqrt{(E_1-E_2)(E_1-E_3)}}{2}, \quad \varkappa_2 = \frac{\vartheta_3(0)}{\vartheta_1(q)} \frac{\sqrt{(E_2-E_1)(E_2-E_3)}}{2},$$

$$\varkappa_3 = \frac{\vartheta_2(0)}{\vartheta_1(q)} \frac{\sqrt{(E_3-E_1)(E_3-E_2)}}{2}, \quad \varkappa_4 = \frac{\vartheta_4(0)}{\vartheta_4(q)} \frac{\sqrt{(E_4-E_2)(E_4-E_3)}}{2}$$

Goal: to make these formula effective

Reality conditions

As a result we have solutions as expressions for symmetric functions in separated variables in terms of generalized theta-functions, but these variables are defined in complex torus. But Liouville torus (and initial physical variables) is real!

Analysis

- As far as coefficients of curve are Hamiltonians which are real, roots of polynomial $Q_{2N+2}(\lambda)$ have to be either pairwise conjugate, either real
- Pairwise conjugated roots correspond to the generic situation (any physically relevant initial condition)
- Real roots restrict strongly the values of Hamiltonians and correspond to non-physical initial conditions

Degeneration

- "colliding of ramification points" $\Rightarrow$ pinching of cycles $\Rightarrow$ reducing of number of independent frequencies in solution
- solitonic solutions

Thank you for attention.