SU(3) Magnet and Generalized Theta-functions of Trigonal Curve

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Outline

1. Dynamical system of isotropic SU(3) magnet
   - Trigonal curve and its integrable systems
   - Integrable system of SU(3) LL equation
   - Quantum model of SU(3) magnet
2. Finite gap integration of the system
   - Separation of variables on generic orbit
   - Generalized Jacobi inverse problem
   - A simple example ($N = 1$)

Trigonal curve and its integrable systems

$$w^3 - w\,\mathcal{I}_{2N}(z) - \mathcal{I}_{3N}(z) = 0$$

$$\Uparrow$$

$$\det\left(\hat{L}(z) - w\right) = 0$$

$$\hat{L} \in \left[\mathfrak{su}(3) \times \mathcal{P}(z, z^{-1})\right]^*, \quad \hat{L}(z) = \hat{L}^{(N)} z^N + \cdots + \hat{L}^{(1)} z + \hat{L}^{(0)}$$

coupled 3-component nonlinear Schrödinger equation

$$\{L_{ij}(u), L_{kl}(v)\} = \mathrm{i}\,\frac{L_{il}(u) - L_{il}(v)}{u-v}\,\delta_{kj} - \mathrm{i}\,\frac{L_{kj}(u) - L_{kj}(v)}{u-v}\delta_{il}$$

Isotropic SU(3) Landau-Lifshits equation

$$\{L_{ij}(u), L_{kl}(v)\} = \mathrm{i}\,\frac{v^N L_{il}(u) - u^N L_{il}(v)}{u-v}\,\delta_{kj} - \mathrm{i}\,\frac{v^N L_{kj}(u) - u^N L_{kj}(v)}{u-v}\delta_{il}$$

Fordy A.P., Kulish P.P, Commun Math Phys, **89** (1983), 427–443.

Integrable system of SU(3) LL equation

$$\hat{L}(z) = \sum_{a=1}^{8} \mu_a(z)\hat{X}_a = \begin{pmatrix} \mu_3(z) + \frac{1}{\sqrt{3}}\mu_8(z) & \mu_1(z) - \mathrm{i}\,\mu_2(z) & \mu_4(z) - \mathrm{i}\,\mu_5(z) \\ \mu_1(z) + \mathrm{i}\,\mu_2(z) & -\mu_3(z) + \frac{1}{\sqrt{3}}\mu_8(z) & \mu_6(z) - \mathrm{i}\,\mu_7(z) \\ \mu_4(z) + \mathrm{i}\,\mu_5(z) & \mu_6(z) + \mathrm{i}\,\mu_7(z) & -\frac{2}{\sqrt{3}}\mu_8(z) \end{pmatrix}$$

$\{\hat{X}_a\}$ is the Gell-Mann basis:

$$[\hat{X}_a, \hat{X}_b] = C_{abc}\hat{X}_c, \qquad \hat{X}_a\hat{X}_b + \hat{X}_b\hat{X}_a = -\tfrac{1}{3}\delta_{ab}\,\mathbb{I} - \tfrac{3}{2}d_{abc}\hat{X}_c.$$

Integrals of motions

$$\mathcal{I}_{2N}(z) = \tfrac{1}{2}\,\mathrm{Tr}\,\hat{L}^2(z) = \textstyle\sum_{\nu=0}^{2N} h_\nu z^\nu, \quad \mathcal{I}_{3N}(z) = \tfrac{1}{3}\,\mathrm{Tr}\,\hat{L}^3(z) = \textstyle\sum_{\nu=0}^{3N} f_\nu z^\nu$$

Equations of the orbit $\mathcal{O}$

$h_0,\, h_1,\, \ldots,\, h_{N-1},$

$f_0,\, f_1,\, \ldots f_{N-1}$

annihilate the Lie-Poisson bracket

Hamiltonians

$h_N,\, h_{N+1},\, \ldots,\, h_{2N-1},$

$f_N,\, f_{N+1},\, \ldots f_{3N-1}$

give rise to nontrivial flows

Coadjoint orbits as phase spaces

Coadjoint orbits of SU(3) group

SU(3) generic orbit $\mathcal{O}^{\mathrm{SU}(3)}_{\mathrm{gen}} = \frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$, $\dim \mathcal{O}^{\mathrm{SU}(3)}_{\mathrm{gen}} = 6$,

SU(3) degenerate orbit $\mathcal{O}^{\mathrm{SU}(3)}_{\mathrm{deg}} = \frac{\mathrm{SU}(3)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$, $\dim \mathcal{O}^{\mathrm{SU}(3)}_{\mathrm{deg}} = 4$.

	generic orbit $\mathcal{O}_{\mathrm{gen}}$	degenerate orbit $\mathcal{O}_{\mathrm{deg}}$
dim	$6N$	$4N$
orbit equations	$h_0 = \mathrm{const}, \ \ldots, \ h_{N-1} = \mathrm{const}$, $f_0 = \mathrm{const}, \ \ldots, \ f_{N-1} = \mathrm{const}$	$\big(\frac{1}{3}\mathcal{I}_{2N}(z)\big)^3 = \big(-\frac{1}{2}\mathcal{I}_{3N}(z)\big)^2$
spectral curve	$w^3 - w\,\mathcal{I}_{2N}(z) - \mathcal{I}_{3N}(z) = 0$ genus $3N-2$	$\big[w \mp 2\big(\frac{1}{3}\,\mathcal{I}_{2N}(z)\big)^{1/2}\big]\times \big[w \pm \big(\frac{1}{3}\,\mathcal{I}_{2N}(z)\big)^{1/2}\big]^2 = 0$
	$\sim$ SU(3) LL equation	$\sim$ LL equation

SU(3) Landau-Lifshits equation

Zero-curvature equation $(h_{N+1}) \quad \dfrac{\partial \mu_a^{(0)}}{\partial t} = \dfrac{\partial \mu_a^{(1)}}{\partial x} \quad (h_N) \quad \Rightarrow$

Generic orbit $\mathcal{O}_{\text{gen}}$ $\qquad \mu_a \equiv \mu_a^{(0)}$

$$\frac{\partial \mu_a}{\partial t} = \frac{2}{4h_0^3 - 27f_0^2} C_{abc} \left(h_0^2 \mu_b \frac{\partial^2 \mu_c}{\partial x^2} - \frac{27}{2} f_0 \left(\mu_b \frac{\partial^2 \eta_c}{\partial x^2} + \eta_b \frac{\partial^2 \mu_c}{\partial x^2} \right) + \right.$$

$$\left. + \frac{27}{4} h_0 \eta_b \frac{\partial^2 \eta_c}{\partial x^2} \right) + \frac{2h_1 h_0^2 - 9 f_1 f_0}{4h_0^3 - 27 f_0^2} \frac{\partial \mu_a}{\partial x} + \frac{9}{2} \frac{2 f_1 h_0 - 3 h_1 f_0}{4h_0^3 - 27 f_0^2} \frac{\partial \eta_a}{\partial x},$$

$$\eta_a = d_{abc} \mu_b \mu_c$$

$$\hat{\mathcal{H}}_{\text{SU(3) LL}} = \frac{1}{4h_0^3 - 27f_0^2} \int \sum\nolimits_a \left(h_0^2 \left(\frac{\partial \mu_a}{\partial x} \right)^2 - 27 f_0 \frac{\partial \mu_a}{\partial x} \frac{\partial \eta_a}{\partial x} + \frac{27}{4} h_0 \left(\frac{\partial \eta_a}{\partial x} \right)^2 \right) dx$$

Degenerate orbit $\mathcal{O}_{\text{deg}}$

$$\frac{\partial \mu_a}{\partial t} = \frac{1}{2h_0} C_{abc} \mu_b \frac{\partial^2 \mu_c}{\partial x^2} + \frac{h_1}{2h_0} \frac{\partial \mu_a}{\partial x}, \qquad \hat{\mathcal{H}}_{\text{LL}} = \frac{1}{4h_0} \int \sum\nolimits_a \left(\frac{\partial \mu_a}{\partial x} \right)^2 dx$$

Bernatska J., Holod P. J Phys A: Math. Theor. 42 (2009) 075401

SU(3) LL equation — classical SU(3) magnet

Quantum model of spin 1 lattice with bilinear-biquadratic Hamiltonian

$$\hat{\mathcal{H}} = -\sum_{n,\delta}\Big\{J\big(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta}\big) + K\big(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta}\big)^2\Big\}$$

$\hat{\boldsymbol{S}}_n = \big(\hat{S}_x^{(n)}, \hat{S}_y^{(n)}, \hat{S}_z^{(n)}\big)$ — spin operators $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ at site n.

Introduction of quadrupole operators

$$\hat{P}_{xy} = \hat{S}_x\hat{S}_y + \hat{S}_y\hat{S}_x, \quad \hat{P}_{xz} = \hat{S}_x\hat{S}_z + \hat{S}_z\hat{S}_x, \quad \hat{P}_{yz} = \hat{S}_y\hat{S}_z + \hat{S}_z\hat{S}_y,$$
$$\hat{P}_{xx} = \hat{S}_x^2, \quad \hat{P}_{yy} = \hat{S}_y^2, \quad \hat{P}_{zz} = \hat{S}_z^2.$$

linearizes the system

$$\hat{\mathcal{H}} = -(J - \tfrac{1}{2}K)\sum_{n,\delta}\big(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta}\big) - \tfrac{1}{2}K\sum_{n,\delta}\big(\hat{\boldsymbol{P}}_n, \hat{\boldsymbol{P}}_{n+\delta}\big),$$

$$\hat{\boldsymbol{P}}_n = \Big(\hat{P}_{xy}^{(n)}, \hat{P}_{xz}^{(n)}, \hat{P}_{yz}^{(n)}, \hat{P}_{xx}^{(n)} - \hat{P}_{yy}^{(n)}, \sqrt{3}\big(\hat{P}_{zz}^{(n)} - \tfrac{2}{3}\mathbb{I}\big)\Big).$$

Mean field approximation

Mean Field Hamiltonian

$$\hat{\mathcal{H}}_{\text{MF}} = -(J - \tfrac{1}{2}K)z \sum_n \big(\hat{\boldsymbol{S}}_n, \langle \hat{\boldsymbol{S}}_n \rangle\big) - \tfrac{1}{2}Kz \sum_n \big(\hat{\boldsymbol{P}}_n, \langle \hat{\boldsymbol{P}}_n \rangle\big)$$

Quasiaverages under the broken symmetry with an external magnetic field

$$\mu_x = \langle \hat{S}_x^{(n)} \rangle, \quad \mu_y = \langle \hat{S}_y^{(n)} \rangle, \quad \mu_z = \langle \hat{S}_z^{(n)} \rangle, \quad \tau_{xy} = \langle \hat{P}_{xy}^{(n)} \rangle, \quad \tau_{xz} = \langle \hat{P}_{xz}^{(n)} \rangle,$$
$$\tau_{yz} = \langle \hat{P}_{yz}^{(n)} \rangle, \quad \tau_{xx} = \langle \hat{P}_{xx}^{(n)} \rangle - \tfrac{2}{3}, \quad \tau_{yy} = \langle \hat{P}_{yy}^{(n)} \rangle - \tfrac{2}{3}, \quad \tau_{zz} = \langle \hat{P}_{zz}^{(n)} \rangle - \tfrac{2}{3},$$

$\tau_{xx} + \tau_{yy} + \tau_{zz} = 0.$

3-dim **representation of the spin algebra:**

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} \\ \mathrm{i} & -\mathrm{i} & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\hat{P}_a$ are tensor operators of weight 1.

Isotropic SU(3) magnet: continuum limit

Isotropic SU(3) magnet ($K = J$)

$$\hat{\mathcal{H}}_{\rm MF} = \tfrac{1}{2}\, rJ \sum_n \sum_a \mu_a \hat{S}_a^{(n)},$$

$$\mathrm{i}\,\hbar \frac{d\hat{S}_a^{(n)}}{dt} = [\hat{S}_a^{(n)}, \hat{\mathcal{H}}_{\rm MF}] \quad \Rightarrow \quad \hbar \frac{\partial \mu_a}{\partial t} = \{\hat{\mathcal{H}}_{\rm eff}, \mu_a\}$$

for $\hat{S}_a^{(n)} \in \{\hat{S}_x^{(n)}, \hat{S}_y^{(n)}, \hat{S}_z^{(n)}, \hat{P}_{xy}^{(n)}, \hat{P}_{xz}^{(n)}, \hat{P}_{yz}^{(n)}, \hat{P}_{xx}^{(n)} - \hat{P}_{yy}^{(n)}, \sqrt{3}\big(\hat{P}_{zz}^{(n)} - \tfrac{2}{3}\,\mathbb{I}\big)\}$.

$$\langle \hat{S}_a^{(n)} \hat{S}_b^{(n)} \rangle \approx \langle \hat{S}_a^{(n)} \rangle \langle \hat{S}_b^{(n)} \rangle \quad \Rightarrow \quad \hbar \frac{\partial \mu_a}{\partial t} = J\ell V_0\, C_{abc} \mu_b \frac{\partial^2 \mu_c}{\partial x^2},$$

ℓ is the lattice distance

V_0 is an infinitesimal volume

$$\hat{\mathcal{H}}_{\rm eff} = \frac{J\ell}{2} \int \sum_a \left(\frac{\partial \mu_a}{\partial x}\right)^2 dx$$

Continuum limit of the isotropic SU(3) magnet coincides with the Landau—Lifshits equation (SU(3) LL equation on a degenerate orbit).

SoV for SU(3) Landau—Lifshits equation

$$\partial_x \hat{L}(z) = [\nabla h_N, \hat{L}(z)], \qquad \partial_t \hat{L}(z) = [\nabla h_{N+1}, \hat{L}(z)],$$

$$\hat{L}(z) = z^N \begin{pmatrix} m + \frac{1}{\sqrt{3}}q & 0 & 0 \\ 0 & -m + \frac{1}{\sqrt{3}}q & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}}q \end{pmatrix} + \\ + \sum_{\nu} z^{\nu} \begin{pmatrix} \mu_3^{(\nu)} + \frac{1}{\sqrt{3}}\mu_8^{(\nu)} & \mu_1^{(\nu)} - \mathrm{i}\,\mu_2^{(\nu)} & \mu_4^{(\nu)} - \mathrm{i}\,\mu_5^{(\nu)} \\ \mu_1^{(\nu)} + \mathrm{i}\,\mu_2^{(\nu)} & -\mu_3^{(\nu)} + \frac{1}{\sqrt{3}}\mu_8^{(\nu)} & \mu_6^{(\nu)} - \mathrm{i}\,\mu_7^{(\nu)} \\ \mu_4^{(\nu)} + \mathrm{i}\,\mu_5^{(\nu)} & \mu_6^{(\nu)} + \mathrm{i}\,\mu_7^{(\nu)} & -\frac{2}{\sqrt{3}}\mu_8^{(\nu)} \end{pmatrix}.$$

$$\dim \mathcal{O}_{\text{gen}} = 6N \quad = \quad 8N \text{ dynamic variables } - 2N \text{ orbit equations}$$

$$\hat{L}(\lambda) = \begin{pmatrix} L_{11}(\lambda) & L_{12}(\lambda) & L_{13}(\lambda) \\ L_{21}(\lambda) & L_{22}(\lambda) & L_{23}(\lambda) \\ L_{31}(\lambda) & L_{32}(\lambda) & L_{33}(\lambda) \end{pmatrix}$$

eliminate from the orbit equations

Parametrization of the orbit

$$h_0 = L^{(0)}_{31} L^{(0)}_{13} + L^{(0)}_{32} L^{(0)}_{23} + L^{(0)}_{21} L^{(0)}_{12} + L^{(0)}_{11} L^{(0)}_{22} + L^{(0)}_{11} L^{(0)}_{33} + L^{(0)}_{22} L^{(0)}_{33}$$

$$f_0 = A^{(0)}_{13} L^{(0)}_{13} + A^{(0)}_{23} L^{(0)}_{23} + A^{(0)}_{33} L^{(0)}_{33}$$

orbit equations

$$h_1 = L^{(1)}_{31} L^{(0)}_{13} + L^{(1)}_{32} L^{(0)}_{23} + L^{(0)}_{31} L^{(1)}_{13} + L^{(0)}_{32} L^{(1)}_{23} + \cdots$$

$$f_1 = A^{(1)}_{13} L^{(0)}_{13} + A^{(1)}_{23} L^{(0)}_{23} + A^{(0)}_{13} L^{(1)}_{13} + A^{(0)}_{23} L^{(1)}_{23} + A^{(1)}_{33} L^{(0)}_{33} + A^{(0)}_{33} L^{(1)}_{33}$$

$\ldots$

$$\Downarrow$$

$$L_{31}(z_k)w_k + A_{13}(z_k) = 0 \qquad L_{32}(z_k)w_k + A_{23}(z_k) = 0$$

$$A_{13}(z_k) \equiv \begin{vmatrix} L_{21}(z_k) & L_{22}(z_k) \\ L_{31}(z_k) & L_{32}(z_k) \end{vmatrix} \qquad A_{23}(z_k) \equiv - \begin{vmatrix} L_{11}(z_k) & L_{12}(z_k) \\ L_{31}(z_k) & L_{32}(z_k) \end{vmatrix}$$

$$\Downarrow$$

$$\mathcal{B}(z_k) \equiv A_{13}(z_k)L_{32}(z_k) - A_{23}(z_k)L_{31}(z_k) = 0$$

consistent equation

$$\mathcal{A}(z_k) \equiv -\frac{A_{13}(z_k)}{L_{31}(z_k)} = -\frac{A_{23}(z_k)}{L_{32}(z_k)} = w_k$$

Separation of variables theorem

Suppose the orbit $\mathcal{O}_{\text{gen}}$ is parameterized by the coordinates

$$\{L_{11}^{(\nu)},\, L_{12}^{(\nu)},\, L_{21}^{(\nu)},\, L_{22}^{(\nu)},\, L_{31}^{(\nu)},\, L_{32}^{(\nu)},\quad \nu=0,\,\dots,\,N-1\}.$$

Then **separation of variables** is realized by the **spectral variables** $\{(z_k, w_k)\colon\ k=1,\,\dots,\,3N\}$ defined by

$$\mathcal{B}(z_k)=0,\qquad w_k=\mathcal{A}(z_k),$$

$$\mathcal{B}(z)=\big[L_{11}(z)-L_{22}(z)\big]L_{31}(z)L_{32}(z)-L_{12}(z)L_{31}^2(z)+L_{21}(z)L_{32}^2(z),$$

$$\mathcal{A}(z_k)=L_{11}(z_k)-\frac{L_{12}(z_k)L_{31}(z_k)}{L_{32}(z_k)}=L_{22}(z_k)-\frac{L_{21}(z_k)L_{32}(z_k)}{L_{31}(z_k)}.$$

the same in
Sklyanin E.K Commun. Math. Phys., **150** (1992) 181–191.

Spectral parametrization of the orbits

$\{(w_k, z_k)\}$ are **quasi-canonically conjugate**:

$$\{z_k, z_l\} = 0, \qquad \{w_k, w_l\} = 0, \qquad \{z_k, w_l\} = -z_k^N \delta_{kl};$$

Liouville 1-form: $\quad \Omega = -\sum_k z_k^{-N} w_k \, dz_k.$

$\{P_k(z_k,\, w_k),\; k=1,\,\dots,\,3N\}$ — **points of the spectral curve**

$$w^3 - w\,\mathcal{I}_{2N}(z) - \mathcal{I}_{3N}(z) = 0, \qquad \text{genus } 3N-2.$$

Phase space is the **generalized Jacobian** of its Riemann surface $\mathcal{R}$

$$\widetilde{\mathrm{Jac}}(\mathcal{R}) = \mathrm{Symm}_{3N} \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{3N}.$$

Previato E. Hyperelliptic quasi-periodic and soliton solution of the nonlinear Schrodinger equation, *Duke Math. J.*, **52** (1985), 323–332.

Spectral & dynamical variables

$$\dim \mathcal{O}_{\text{gen}} = 6N \quad \Rightarrow \quad 3N \text{ points } (z_k, w_k) \text{ are required.}$$

To obtain the polynomial $\mathcal{B}$ of degree $3N$

$$R^{-1}\hat{L}(z)R = z^N \begin{pmatrix} \frac{q}{\sqrt{3}} & \frac{m}{\sqrt{2}} & \frac{m}{\sqrt{2}} \\ \frac{m}{\sqrt{2}} & -\frac{q}{2\sqrt{3}} & \frac{\sqrt{3}q}{2} \\ \frac{m}{\sqrt{2}} & \frac{\sqrt{3}q}{2} & -\frac{q}{2\sqrt{3}} \end{pmatrix} + \cdots, \quad R = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

$$\mathcal{B}(z) = -\frac{m(m^2-3q^2)}{2\sqrt{2}} z^{3N} + \cdots = \mathcal{B}_{3N} \prod_{k=1}^{3N} (z - z_k).$$

The equation for $L_{31}^{(\nu)}$, $A_{13}^{(\nu)}$, $A_{13}^{(\nu+N)}$, $L_{32}^{(\nu)}$, $A_{23}^{(\nu)}$, $A_{23}^{(\nu+N)}$, $\nu = 0, \ldots, N-1$

$$L_{31}(z_k)w_k + A_{13}(z_k) = 0 \qquad L_{32}(z_k)w_k + A_{23}(z_k) = 0$$

Equations for the spectral variables

From the stationary (h_N) and the evolutionary (h_{N+1}) flows

$$\frac{dz_k}{dx} = \frac{\left[L_{32}^{(0)} L_{31}(z_k) - L_{31}^{(0)} L_{32}(z_k)\right] \frac{\partial}{\partial w} P(z_k, w)}{\mathcal{B}_{3N} z_k \prod_j' (z_k - z_j)},$$

$$\frac{dz_k}{dt} = \frac{\left[\left(L_{32}^{(0)} + L_{32}^{(1)} z_k\right) L_{31}(z_k) - \left(L_{31}^{(0)} + L_{31}^{(1)} z_k\right) L_{32}(z_k)\right] \frac{\partial}{\partial w} P(z_k, w)}{\mathcal{B}_{3N} z_k^2 \prod_j' (z_k - z_j)}$$

for (w_k, z_k) satisfying $w_k^2 + L_{22}(z_k) w_k + A_{33}(z_k) = 0$.

At every point (w_k, z_k) the spectral curve is reduced to

$$\left[w_k - L_{33}(z_k)\right]\left[w_k^2 + L_{22}(z_k) w_k + A_{33}(z_k)\right] = 0.$$

Abelian differentials

Holomorphic differentials

$$\phi_\nu = \frac{z^{\nu-1}\,dz}{\frac{\partial}{\partial w}P(w,z)}, \qquad \nu = 1,\,2,\,\ldots,\,2N-1,$$

$$\phi_{\nu+2N} = \frac{wz^{\nu}\,dz}{\frac{\partial}{\partial w}P(w,z)}, \qquad \nu = 1,\,2,\,\ldots,\,N-1.$$

Abelian differentials of the third kind

$$\Phi_z = \frac{z^{2N-1}\,dz}{\frac{\partial}{\partial w}P(w,z)}, \qquad \Phi_w = \frac{wz^{N-1}\,dz}{\frac{\partial}{\partial w}P(w,z)}.$$

$$w^3 = w\mathcal{I}_{2N}(z) + \mathcal{I}_{3N}(z), \qquad z \mapsto \tau = 1/z, \quad w \mapsto \zeta = w/z^N$$

$$\tau \to 0: \quad \zeta^3 = h_{2N}\zeta + f_{3N} = 0 \quad \Rightarrow \quad \zeta_1,\,\zeta_2,\,\zeta_3$$

$$\operatorname{res}_{\infty_k}\Phi_z = -\frac{1}{3\zeta_k^2 - h_{2N}}, \qquad \operatorname{res}_{\infty_k}\Phi_w = -\frac{\zeta_k}{3\zeta_k^2 - h_{2N}}.$$

Extended canonical differential basis

For canonical homology basis $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_{3N-2}, \mathfrak{b}_1, \ldots, \mathfrak{b}_{3N-2}\}$ on $\widetilde{\mathrm{Jac}}(\mathcal{R})$

$$\omega = A^{-1}\phi, \qquad A = (A_{ij}), \quad A_{ij} = \int_{\mathfrak{a}_j} \phi_i, \quad B = (B_{ij}), \quad B_{ij} = \int_{\mathfrak{b}_j} \phi_i$$

$$\Omega_z = \Phi_z - \sum_{i=1}^{3N-2} A_{zi}\omega_i, \qquad A_{zi} = \int_{\mathfrak{a}_j} \Phi_z$$

$$\Omega_w = \Phi_w - \sum_{i=1}^{3N-2} A_{wi}\omega_i, \qquad A_{wi} = \int_{\mathfrak{a}_j} \Phi_w.$$

$$\Omega_{21} = \frac{3\zeta_1^2 - h_{2N}}{\zeta_1 - \zeta_3}\Big(\Omega_w - \zeta_3\Omega_z\Big), \qquad \mathrm{res}_{\infty_2}\,\Omega_{21} = 1, \quad \mathrm{res}_{\infty_1}\,\Omega_{21} = -1;$$

$$\Omega_{31} = \frac{3\zeta_1^2 - h_{2N}}{\zeta_1 - \zeta_2}\Big(\Omega_w - \zeta_2\Omega_z\Big), \qquad \mathrm{res}_{\infty_3}\,\Omega_{31} = 1, \quad \mathrm{res}_{\infty_1}\,\Omega_{31} = -1.$$

Braden H. W., Fedorov Y. N., Journal of Geometry and Physics, **58** (2008), 1346–1354

Generalized Jacobi inverse problem

$$\xi = \sum_{k=1}^{3N} \int_{P_0}^{P_k} \phi \qquad\qquad \boldsymbol{u} = \sum_{k=1}^{3N} \int_{P_0}^{P_k} \boldsymbol{\omega} = \sum_{k=1}^{3N} \mathcal{A}(P_k)$$

$$\xi_z = \sum_{k=1}^{3N} \int_{P_0}^{P_k} \Phi_z \qquad \Rightarrow \qquad U_{21} = \sum_{k=1}^{3N} \int_{P_0}^{P_k} \Omega_{21}$$

$$\xi_w = \sum_{k=1}^{3N} \int_{P_0}^{P_k} \Phi_w \qquad\qquad U_{31} = \sum_{k=1}^{3N} \int_{P_0}^{P_k} \Omega_{31}$$

$\widehat{\boldsymbol{u}} = (\boldsymbol{u},\, U_{21},\, U_{31}), \qquad \widehat{\boldsymbol{\omega}} = (\boldsymbol{\omega},\, \Omega_{21},\, \Omega_{31})$

$$\widehat{\boldsymbol{u}} = \sum_{k=1}^{3N} \int_{P_0}^{P_k} \widehat{\boldsymbol{\omega}} = \sum_{k=1}^{3N} \widehat{\mathcal{A}}(P_k)$$

Generalized Theta-function

$$\Theta(\widehat{\boldsymbol{u}}) = e^{U_{31}-\mathcal{K}_3-\Delta_3}\theta\big(\boldsymbol{u}-\boldsymbol{K}-\boldsymbol{\mathcal{A}}(\infty_1)-\boldsymbol{\mathcal{A}}(\infty_2)\big)+ \\ + e^{U_{21}-\mathcal{K}_2-\Delta_2}\theta\big(\boldsymbol{u}-\boldsymbol{K}-\boldsymbol{\mathcal{A}}(\infty_1)-\boldsymbol{\mathcal{A}}(\infty_3)\big)- \\ -\theta\big(\boldsymbol{u}-\boldsymbol{K}-\boldsymbol{\mathcal{A}}(\infty_2)-\boldsymbol{\mathcal{A}}(\infty_3)\big),$$

$\boldsymbol{K}$ — Riemann constants.

$$\theta(\boldsymbol{u}) \equiv \theta(\boldsymbol{u}\,|\,B) = \sum_{\boldsymbol{n}\in\mathbb{Z}^{3N-2}} \exp\big\{2\,\mathrm{i}\,\pi\boldsymbol{u}\cdot\boldsymbol{n}+\mathrm{i}\,\pi\boldsymbol{n}\cdot B\boldsymbol{n}\big\},$$

$$\Delta_2 = \int_{P_0}^{\infty_3}\Omega_{21}, \qquad \mathcal{K}_2 = \sum_{i=1}^{3N-2}\oint_{\mathfrak{a}_i}\left(\omega_i(P)\int_{P_0}^{P}\Omega_{21}\right)+\oint_{\mathfrak{b}_i}\Omega_{21},$$

$$\Delta_3 = \int_{P_0}^{\infty_2}\Omega_{31}, \qquad \mathcal{K}_3 = \sum_{i=1}^{3N-2}\oint_{\mathfrak{a}_i}\left(\omega_i(P)\int_{P_0}^{P}\Omega_{31}\right)+\oint_{\mathfrak{b}_i}\Omega_{31}.$$

A simple example ($N = 1$)

The stationary flow

$$\frac{d}{dx} z_1 = -\frac{(z_2 - z_3)}{\det W}\left[3w_1^2 - \mathcal{I}_2(z_1)\right],$$
$$\frac{d}{dx} z_2 = -\frac{(z_3 - z_1)}{\det W}\left[3w_2^2 - \mathcal{I}_2(z_2)\right],$$
$$\frac{d}{dx} z_3 = -\frac{(z_1 - z_2)}{\det W}\left[3w_3^2 - \mathcal{I}_2(z_3)\right]$$

$$\det W = \begin{vmatrix} w_1 & z_1 & 1 \\ w_2 & z_2 & 1 \\ w_3 & z_3 & 1 \end{vmatrix} =$$
$$= w_1(z_2 - z_3) + w_2(z_3 - z_1) +$$
$$+ w_3(z_1 - z_2).$$

Generalized Jacobi inverse problem

The map $(z_1, z_2, z_3) \mapsto (\xi, \xi_z, \xi_w)$ is nonsingular if $\det W \neq 0$

$$\xi = \sum_{k=1}^{3} \int_{z_0}^{z_k} \frac{dz}{3w^2 - \mathcal{I}_2(z)}, \quad \xi_z = \sum_{k=1}^{3} \int_{z_0}^{z_k} \frac{z dz}{3w^2 - \mathcal{I}_2(z)},$$

$$\xi_w = \sum_{k=1}^{3} \int_{z_0}^{z_k} \frac{w\, dz}{3w^2 - \mathcal{I}_2(z)};$$

$$\xi = C, \quad \xi_z = C_z, \quad \xi_w = C_w - x.$$

Spectral curve parametrization ($N = 1$)

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The end

Conclusion and Discussion

- We propose an effective way how to realize a scheme of variable separating for algebra $\mathfrak{sl}(3)$.
  An orbit approach is essentially used in computations.
- It becomes clear, that separation of variables after Sklyanin is applicable to systems on generic orbits,
  and requires an adaptation to degenerate orbits.
- Dynamical systems of the kind (on generic orbits of the $\mathfrak{sl}(3)$ loop algebra) need two additional Abelian differentials for integration.